## Stability issues for the conformal method in the focusing case Three days workshop in Mathematical General Relativity – Lyon

Bruno Premoselli

Université Libre de Bruxelles (ULB)

April 16, 2019

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 $T_{\alpha\beta}X^{\alpha}X^{\beta} \geq 0$  for any timelike field X,

which says that the observed local matter density is always nonnegative.

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• Klein-Gordon fields:  $V(\Psi) = \frac{1}{2}m\Psi^2$ , m > 0

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#### $\partial_t g_t \sim 2K_t.$

**Question:** is any choice (g, K) on  $M^n \times \{0\}$  admissible to be induced by a solution of the Einstein equations?

The structure  $\mathcal{M}^{n+1} = (\mathcal{M}^n \times \mathbb{R}, h)$  induces necessary conditions on  $g := g_0$  and  $\mathcal{K} := \mathcal{K}_0$  to solve the Einstein equations:

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We now show how these necessary conditions are obtained.

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The structure  $\mathcal{M}^{n+1} = (\mathcal{M}^n \times \mathbb{R}, h)$  induces necessary conditions on  $g := g_0$  and  $K := K_0$  to solve the Einstein equations:

$$Ric(h)-rac{1}{2}R(h)h=T.$$

We now show how these necessary conditions are obtained. Let  $\vec{n}$  be a unit normal vector to  $M^n \times \{0\}$  in  $\mathcal{M}^{n+1}$ , represented by the index 0, and  $(e_i)_i$  an orthonormal basis on  $M^n$ .

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An analogous computation using the Gauss equations shows that if  $(M^n \times \mathbb{R}, h)$  solves the Einstein equations, then  $g = h_{|M^n|}$  and K satisfy the so-called constraint equations:

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Solving (C) therefore determines the admissible initial-data sets for the Einstein equations: we'll focus on its resolution and on the properties of **some** of its solutions in the rest of this talk.

Let  $(M^n, g)$  be a Riemannian manifold and consider in  $M^n$ :

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We won't dwell on other techniques, e.g. glueing methods, which proved extremely useful to construct solutions of (C) in a variety of geometric settings (see e.g. Chruściel-Isenberg-Pollack '05, Corvino-Schoen '06, Carlotto-Schoen '15, Chruściel-Delay '18, etc...).

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The conformal method was initiated by Lichnerowicz ('48) in the maximal case  $\tau \equiv \mathbf{0}$ 

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### The scalar constraint equation I

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Transforming in the same way the vector equation shows that an initial-data set

$$(g, K, \rho, J) = \left(u^{\frac{4}{n-2}}g_0, \frac{\tau}{n}u^{\frac{4}{n-2}}g_0 + u^{-2}\left(\sigma + \mathcal{L}_{g_0}W\right), \psi, u^{-\frac{2n}{n-2}}\pi\right)$$
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# Contents

#### The initial-value formulation of the Einstein equations

- Initial-data sets and well-posedness
- The conformal method
- The Einstein-Lichnerowicz system

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- The setting
- Existence theory
- Definition of Elliptic stability

#### 3 Elliptic Stability: the results

- The defocusing case
- Elliptic stability in the focusing case
- The proof in the focusing case

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These two regimes exhibit a very different behavior analytically speaking.

An illustration: existence theory in the decoupled case Under the assumption  $\nabla \tau \equiv 0$ 

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- When f(τ, ψ, V) > 0 (focusing), the first existence result is in Hebey-Pacard-Pollack ('07). Solutions only exist for a limited set of coefficients, and there are generically two solutions (P., '14). Their bifurcation diagram can be complex, with multiple pitchfork bifurcations (Bizón-Pletka-Simon '15, Chruściel-Gicquaud '17) and in pathological cases the equation can have an infinite number of (non-compact) solutions (P.-Wei, '15).

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 for the defocusing case f ≤ 0: Maxwell ('04, '08), Holst-Nagy-Tsogtgerel ('08), Dahl-Gicquaud-Humbert ('13)

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We now introduce a notion of stability to answer these questions and investigate the "well-posedness" of the conformal method.

Recall that if  $\mathcal{D} = (V, \psi, \pi, \tau, \sigma)$  are physics data, the Einstein-Lichnerowicz system with physics data  $\mathcal{D}$  is:

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In the second lecture we will state and prove some stability results.

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Thank you for your attention! (End of Lecture 1 and coffe break for everyone)

# Contents

#### The initial-value formulation of the Einstein equations

- Initial-data sets and well-posedness
- The conformal method
- The Einstein-Lichnerowicz system

#### 2 The Einstein-Lichnerowicz system in a scalar-field theory

- The setting
- Existence theory
- Definition of Elliptic stability

#### 3 Elliptic Stability: the results

- The defocusing case
- Elliptic stability in the focusing case
- The proof in the focusing case

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The statement of the stability results (and their proofs) again heavily depend on the focusing or defocusing setting.

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Then the set of solutions of the vacuum Einstein-Lichnerowicz system is non-empty and compact in  $C^{2}(M)$ .

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### Elliptic Stabiliy: the results in the focusing case

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Let  $(M^n, g)$  be a closed locally conformally flat manifold of dimension  $n \ge 3$ . Let  $\mathcal{D} = (V, \psi, \pi, \tau, \sigma)$  be some focusing physics data, i.e. with  $f(\tau, \psi, V) > 0$ .

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The proof proceeds by contradiction. Let  $(u_{\alpha}, W_{\alpha})_{\alpha}$  be a sequence of solutions of the system and assume that it blows-up:

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$$\mu_{\alpha}^{-n} := \max_{M} \left( u_{\alpha}^{\frac{2n}{n-2}} + |\mathcal{L}_{g}W_{\alpha}|_{g} \right).$$

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If  $x_{\alpha}$  is the point where this maximum is attained, we let, for  $x \in \mathbb{R}^n$ :

 $v_{\alpha}(x) = \mu_{\alpha}^{\frac{n-2}{2}} \exp_{x_{\alpha}}^{*} u_{\alpha}(\mu_{\alpha}x) \quad \text{and} \quad X_{\alpha}(x) = \mu_{\alpha}^{n-1} \exp_{x_{\alpha}}^{*} W_{\alpha}(\mu_{\alpha}x)$ 

be the rescalings of  $u_{\alpha}$  and  $W_{\alpha}$  at distances comparable to  $\mu_{\alpha}$  to  $x_{\alpha}$ .

$$ig( riangle_{\xi} U = f(x_0) U^{2^*-1} \ imes \overrightarrow{\Delta_g} X = 0.$$

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 $( \widecheck{\bigtriangleup_g} X = 0. )$ 

where  $x_0 = \lim x_{\alpha}$ .

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- for any  $x \in M$ :

$$\left(\min_{i=1,\ldots,N_{\alpha}}d_{g}\left(x_{i,\alpha},x\right)\right)^{n}\left(u_{\alpha}(x)^{2^{*}}+\left|\mathcal{L}_{g}W_{\alpha}\right|_{g}(x)\right)\leq C.$$

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Denote by  $\rho_{i,\alpha}$  the maximum radius around which the concentration point  $x_{i,\alpha}$  is dominant.

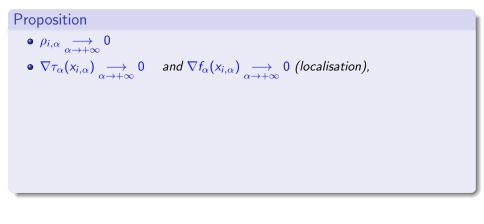
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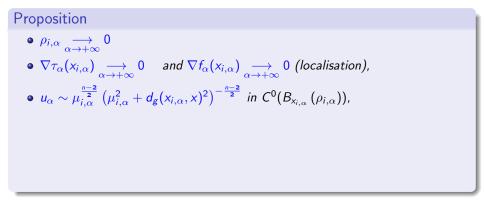


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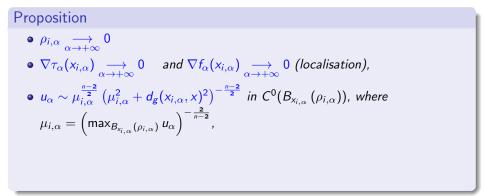
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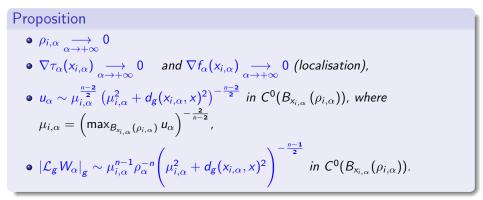
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- It was used to prove the generic existence of two solutions of the Einstein-Lichnerowicz equation in the focusing case (P. '14)
- It has very recently been used to prove involved existence results in the non-variational setting of the so-called volumetric drift method (Vâlcu, '19).

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Thank you for your attention!

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