Stability issues for the conformal method in the focusing case

Three days workshop in Mathematical General Relativity – Lyon

Bruno Premoselli

Université Libre de Bruxelles (ULB)

April 16, 2019

Contents

The Einstein equations

(Mathematical) General Relativity is concerned with investigating the properties of spacetimes.

The Einstein equations

(Mathematical) General Relativity is concerned with investigating the properties of spacetimes. They are $n+1$-dimensional Lorentzian manifolds $(\mathcal{M}^{n+1}, h)$, $n \geq 3$, satisfying the Einstein equations:

The Einstein equations

(Mathematical) General Relativity is concerned with investigating the properties of spacetimes. They are $n+1$-dimensional Lorentzian manifolds $(\mathcal{M}^{n+1}, h)$, $n \geq 3$, satisfying the Einstein equations:

$$Ric(h) - \frac{1}{2}R(h)h = T. \qquad (E)$$

The Einstein equations

(Mathematical) General Relativity is concerned with investigating the properties of spacetimes. They are $n+1$-dimensional Lorentzian manifolds $(\mathcal{M}^{n+1}, h)$, $n \geq 3$, satisfying the Einstein equations:

$$Ric(h) - \frac{1}{2} R(h) h = T. \qquad (E)$$

Here $\mathrm{Ric}(h)$ is the Ricci tensor of h, $R(h)$ is its scalar curvature and T is the stress-energy tensor.

The Einstein equations

(Mathematical) General Relativity is concerned with investigating the properties of spacetimes. They are $n+1$-dimensional Lorentzian manifolds $(\mathcal{M}^{n+1}, h)$, $n \geq 3$, satisfying the Einstein equations:

$$Ric(h) - \frac{1}{2} R(h) h = T. \qquad (E)$$

Here $\mathrm{Ric}(h)$ is the Ricci tensor of h, $R(h)$ is its scalar curvature and T is the stress-energy tensor.

T describes the coupling between gravity and matter/scalar fields in $\mathcal{M}$.

The Einstein equations

(Mathematical) General Relativity is concerned with investigating the properties of spacetimes. They are $n+1$-dimensional Lorentzian manifolds $(\mathcal{M}^{n+1}, h)$, $n \geq 3$, satisfying the Einstein equations:

$$Ric(h) - \frac{1}{2} R(h) h = T. \qquad (E)$$

Here $\mathrm{Ric}(h)$ is the Ricci tensor of h, $R(h)$ is its scalar curvature and T is the stress-energy tensor.

T describes the coupling between gravity and matter/scalar fields in $\mathcal{M}$. It is divergence-free:

$$\nabla^\alpha T_{\alpha\beta} = 0,$$

The Einstein equations

(Mathematical) General Relativity is concerned with investigating the properties of spacetimes. They are $n+1$-dimensional Lorentzian manifolds $(\mathcal{M}^{n+1}, h)$, $n \geq 3$, satisfying the Einstein equations:

$$Ric(h) - \frac{1}{2} R(h) h = T. \qquad (E)$$

Here $\mathrm{Ric}(h)$ is the Ricci tensor of h, $R(h)$ is its scalar curvature and T is the stress-energy tensor.

T describes the coupling between gravity and matter/scalar fields in $\mathcal{M}$. It is divergence-free:

$$\nabla^\alpha T_{\alpha\beta} = 0,$$

and is usually required to satisfy some form of energy condition. For instance the weak energy condition:

The Einstein equations

(Mathematical) General Relativity is concerned with investigating the properties of spacetimes. They are $n+1$-dimensional Lorentzian manifolds $(\mathcal{M}^{n+1}, h)$, $n \geq 3$, satisfying the Einstein equations:

$$Ric(h) - \frac{1}{2} R(h) h = T. \qquad (E)$$

Here $\mathrm{Ric}(h)$ is the Ricci tensor of h, $R(h)$ is its scalar curvature and T is the stress-energy tensor.

T describes the coupling between gravity and matter/scalar fields in $\mathcal{M}$. It is divergence-free:

$$\nabla^\alpha T_{\alpha\beta} = 0,$$

and is usually required to satisfy some form of energy condition. For instance the weak energy condition:

$$T_{\alpha\beta} X^\alpha X^\beta \geq 0 \quad \text{for any timelike field } X,$$

which says that the observed local matter density is always nonnegative.

An example: the scalar-field setting

In a scalar-field setting (our main focus) gravity is coupled to matter via a scalar-field.

An example: the scalar-field setting

In a scalar-field setting (our main focus) gravity is coupled to matter via a scalar-field. Here T is given by:

An example: the scalar-field setting

In a scalar-field setting (our main focus) gravity is coupled to matter via a scalar-field. Here T is given by:

$$T_{\alpha\beta} = \nabla_\alpha \Psi \nabla_\beta \Psi - \left(\frac{1}{2} |\nabla \Psi|_h^2 + V(\Psi) \right) h_{\alpha\beta}.$$

An example: the scalar-field setting

In a scalar-field setting (our main focus) gravity is coupled to matter via a scalar-field. Here T is given by:

$$T_{\alpha\beta} = \nabla_\alpha \Psi \nabla_\beta \Psi - \left(\frac{1}{2} |\nabla \Psi|_h^2 + V(\Psi) \right) h_{\alpha\beta}.$$

Here $\Psi \in C^\infty(\mathcal{M}^{n+1})$ is a scalar-field

An example: the scalar-field setting

In a scalar-field setting (our main focus) gravity is coupled to matter via a scalar-field. Here T is given by:

$$T_{\alpha\beta} = \nabla_\alpha \Psi \nabla_\beta \Psi - \left(\frac{1}{2} |\nabla \Psi|_h^2 + V(\Psi) \right) h_{\alpha\beta}.$$

Here $\Psi \in C^\infty(\mathcal{M}^{n+1})$ is a scalar-field and $V \in C^\infty(\mathbb{R})$ is its potential.

An example: the scalar-field setting

In a scalar-field setting (our main focus) gravity is coupled to matter via a scalar-field. Here T is given by:

$$T_{\alpha\beta} = \nabla_\alpha \Psi \nabla_\beta \Psi - \left(\frac{1}{2} |\nabla \Psi|_h^2 + V(\Psi) \right) h_{\alpha\beta}.$$

Here $\Psi \in C^\infty(\mathcal{M}^{n+1})$ is a scalar-field and $V \in C^\infty(\mathbb{R})$ is its potential. They satisfy in addition:

$$\Box_h \Psi = V'(\Psi).$$

An example: the scalar-field setting

In a scalar-field setting (our main focus) gravity is coupled to matter via a scalar-field. Here T is given by:

$$T_{\alpha\beta} = \nabla_\alpha \Psi \nabla_\beta \Psi - \left(\frac{1}{2} |\nabla \Psi|_h^2 + V(\Psi) \right) h_{\alpha\beta}.$$

Here $\Psi \in C^\infty(\mathcal{M}^{n+1})$ is a scalar-field and $V \in C^\infty(\mathbb{R})$ is its potential. They satisfy in addition:

$$\Box_h \Psi = V'(\Psi).$$

Ψ describes the distribution of matter and/or energy in the universe.

An example: the scalar-field setting

In a scalar-field setting (our main focus) gravity is coupled to matter via a scalar-field. Here T is given by:

$$T_{\alpha\beta} = \nabla_\alpha \Psi \nabla_\beta \Psi - \left(\frac{1}{2} |\nabla \Psi|_h^2 + V(\Psi) \right) h_{\alpha\beta}.$$

Here $\Psi \in C^\infty(\mathcal{M}^{n+1})$ is a scalar-field and $V \in C^\infty(\mathbb{R})$ is its potential. They satisfy in addition:

$$\Box_h \Psi = V'(\Psi).$$

Ψ describes the distribution of matter and/or energy in the universe.

Physical cases covered by this model:

- The vacuum case $\Psi \equiv 0$,

An example: the scalar-field setting

In a scalar-field setting (our main focus) gravity is coupled to matter via a scalar-field. Here T is given by:

$$T_{\alpha\beta} = \nabla_\alpha \Psi \nabla_\beta \Psi - \left(\frac{1}{2} |\nabla \Psi|_h^2 + V(\Psi) \right) h_{\alpha\beta}.$$

Here $\Psi \in C^\infty(\mathcal{M}^{n+1})$ is a scalar-field and $V \in C^\infty(\mathbb{R})$ is its potential. They satisfy in addition:

$$\Box_h \Psi = V'(\Psi).$$

Ψ describes the distribution of matter and/or energy in the universe.

Physical cases covered by this model:

- The vacuum case $\Psi \equiv 0$, with or without cosmological constant: $V \equiv \Lambda$.

An example: the scalar-field setting

In a scalar-field setting (our main focus) gravity is coupled to matter via a scalar-field. Here T is given by:

$$T_{\alpha\beta} = \nabla_\alpha \Psi \nabla_\beta \Psi - \left(\frac{1}{2} |\nabla \Psi|_h^2 + V(\Psi) \right) h_{\alpha\beta}.$$

Here $\Psi \in C^\infty(\mathcal{M}^{n+1})$ is a scalar-field and $V \in C^\infty(\mathbb{R})$ is its potential. They satisfy in addition:

$$\Box_h \Psi = V'(\Psi).$$

Ψ describes the distribution of matter and/or energy in the universe.

Physical cases covered by this model:

- The vacuum case $\Psi \equiv 0$, with or without cosmological constant: $V \equiv \Lambda$. In this case the Einstein equations become:

$$\mathrm{Ric}(h) = \frac{2\Lambda}{n-1} h.$$

An example: the scalar-field setting

In a scalar-field setting (our main focus) gravity is coupled to matter via a scalar-field. Here T is given by:

$$T_{\alpha\beta} = \nabla_\alpha \Psi \nabla_\beta \Psi - \left(\frac{1}{2} |\nabla \Psi|_h^2 + V(\Psi) \right) h_{\alpha\beta}.$$

Here $\Psi \in C^\infty(\mathcal{M}^{n+1})$ is a scalar-field and $V \in C^\infty(\mathbb{R})$ is its potential. They satisfy in addition:

$$\Box_h \Psi = V'(\Psi).$$

Ψ describes the distribution of matter and/or energy in the universe.

Physical cases covered by this model:

- The vacuum case $\Psi \equiv 0$, with or without cosmological constant: $V \equiv \Lambda$. In this case the Einstein equations become:

$$\mathrm{Ric}(h) = \frac{2\Lambda}{n-1} h.$$

- Klein-Gordon fields: $V(\Psi) = \frac{1}{2} m \Psi^2$, $m > 0$

The evolution problem

Physically realistic spacetimes need to possess good causal properties.

The evolution problem

Physically realistic spacetimes need to possess good causal properties.

One looks for spacetimes which split as $\mathcal{M}^{n+1} = M^n \times \mathbb{R}$,

The evolution problem

Physically realistic spacetimes need to possess good causal properties.

One looks for spacetimes which split as $\mathcal{M}^{n+1} = M^n \times \mathbb{R}$, where each $M^n \times \{t\}$ is a spacelike hypersurface, that is

The evolution problem

Physically realistic spacetimes need to possess good causal properties.

One looks for spacetimes which split as $\mathcal{M}^{n+1} = M^n \times \mathbb{R}$, where each $M^n \times \{t\}$ is a spacelike hypersurface, that is h restricted to $M^n \times \{t\}$ is Riemannian.

The evolution problem

Physically realistic spacetimes need to possess good causal properties.

One looks for spacetimes which split as $\mathcal{M}^{n+1} = M^n \times \mathbb{R}$, where each $M^n \times \{t\}$ is a spacelike hypersurface, that is h restricted to $M^n \times \{t\}$ is Riemannian. (The real notion: globally hyperbolic spacetimes.)

The evolution problem

Physically realistic spacetimes need to possess good causal properties.

One looks for spacetimes which split as $\mathcal{M}^{n+1} = M^n \times \mathbb{R}$, where each $M^n \times \{t\}$ is a spacelike hypersurface, that is h restricted to $M^n \times \{t\}$ is Riemannian. (The real notion: globally hyperbolic spacetimes.)

Let g_t and K_t be the metric and second fundamental form of $M^n \times \{t\} \subset \mathcal{M}^{n+1}$.

The evolution problem

Physically realistic spacetimes need to possess good causal properties.

One looks for spacetimes which split as $\mathcal{M}^{n+1} = M^n \times \mathbb{R}$, where each $M^n \times \{t\}$ is a spacelike hypersurface, that is h restricted to $M^n \times \{t\}$ is Riemannian. (The real notion: globally hyperbolic spacetimes.)

Let g_t and K_t be the metric and second fundamental form of $M^n \times \{t\} \subset \mathcal{M}^{n+1}$. The Lorentzian metric h can be thus seen as a path $t \mapsto g_t$ of Riemannian metrics that evolve "in time",

The evolution problem

Physically realistic spacetimes need to possess good causal properties.

One looks for spacetimes which split as $\mathcal{M}^{n+1} = M^n \times \mathbb{R}$, where each $M^n \times \{t\}$ is a spacelike hypersurface, that is h restricted to $M^n \times \{t\}$ is Riemannian. (The real notion: globally hyperbolic spacetimes.)

Let g_t and K_t be the metric and second fundamental form of $M^n \times \{t\} \subset \mathcal{M}^{n+1}$. The Lorentzian metric h can be thus seen as a path $t \mapsto g_t$ of Riemannian metrics that evolve "in time", and whose time derivative is:

The evolution problem

Physically realistic spacetimes need to possess good causal properties.

One looks for spacetimes which split as $\mathcal{M}^{n+1} = M^n \times \mathbb{R}$, where each $M^n \times \{t\}$ is a spacelike hypersurface, that is h restricted to $M^n \times \{t\}$ is Riemannian. (The real notion: globally hyperbolic spacetimes.)

Let g_t and K_t be the metric and second fundamental form of $M^n \times \{t\} \subset \mathcal{M}^{n+1}$. The Lorentzian metric h can be thus seen as a path $t \mapsto g_t$ of Riemannian metrics that evolve "in time", and whose time derivative is:

$$\partial_t g_t \sim 2K_t.$$

The evolution problem

Physically realistic spacetimes need to possess good causal properties.

One looks for spacetimes which split as $\mathcal{M}^{n+1} = M^n \times \mathbb{R}$, where each $M^n \times \{t\}$ is a spacelike hypersurface, that is h restricted to $M^n \times \{t\}$ is Riemannian. (The real notion: globally hyperbolic spacetimes.)

Let g_t and K_t be the metric and second fundamental form of $M^n \times \{t\} \subset \mathcal{M}^{n+1}$. The Lorentzian metric h can be thus seen as a path $t \mapsto g_t$ of Riemannian metrics that evolve "in time", and whose time derivative is:

$$\partial_t g_t \sim 2K_t.$$

Question: is any choice (g, K) on $M^n \times \{0\}$ admissible to be induced by a solution of the Einstein equations?

The constraint equations

The constraint equations

The structure $\mathcal{M}^{n+1} = (M^n \times \mathbb{R}, h)$ induces necessary conditions on $g := g_0$ and $K := K_0$ to solve the Einstein equations:

The constraint equations

The structure $\mathcal{M}^{n+1} = (M^n \times \mathbb{R}, h)$ induces necessary conditions on $g := g_0$ and $K := K_0$ to solve the Einstein equations:

$$Ric(h) - \frac{1}{2} R(h) h = T.$$

The constraint equations

The structure $\mathcal{M}^{n+1} = (M^n \times \mathbb{R}, h)$ induces necessary conditions on $g := g_0$ and $K := K_0$ to solve the Einstein equations:

$$Ric(h) - \frac{1}{2}R(h)h = T.$$

We now show how these necessary conditions are obtained.

The constraint equations

The structure $\mathcal{M}^{n+1} = (M^n \times \mathbb{R}, h)$ induces necessary conditions on $g := g_0$ and $K := K_0$ to solve the Einstein equations:

$$Ric(h) - \frac{1}{2}R(h)h = T.$$

We now show how these necessary conditions are obtained. Let $\vec{n}$ be a unit normal vector to $M^n \times \{0\}$ in $\mathcal{M}^{n+1}$, represented by the index 0, and $(e_i)_i$ an orthonormal basis on M^n.

The constraint equations

The structure $\mathcal{M}^{n+1} = (M^n \times \mathbb{R}, h)$ induces necessary conditions on $g := g_0$ and $K := K_0$ to solve the Einstein equations:

$$Ric(h) - \frac{1}{2}R(h)h = T.$$

We now show how these necessary conditions are obtained. Let $\vec{n}$ be a unit normal vector to $M^n \times \{0\}$ in $\mathcal{M}^{n+1}$, represented by the index 0, and $(e_i)_i$ an orthonormal basis on M^n. The Einstein equations yield:

$$T_{i0} = \mathrm{Ric}_{i0}$$

The constraint equations

The structure $\mathcal{M}^{n+1} = (M^n \times \mathbb{R}, h)$ induces necessary conditions on $g := g_0$ and $K := K_0$ to solve the Einstein equations:

$$Ric(h) - \frac{1}{2}R(h)h = T.$$

We now show how these necessary conditions are obtained. Let $\vec{n}$ be a unit normal vector to $M^n \times \{0\}$ in $\mathcal{M}^{n+1}$, represented by the index 0, and $(e_i)_i$ an orthonormal basis on M^n. The Einstein equations yield:

$$T_{i0} = \mathrm{Ric}_{i0} = h^{\alpha\beta} Rm(h)_{\alpha i 0 \beta}$$

The constraint equations

The structure $\mathcal{M}^{n+1} = (M^n \times \mathbb{R}, h)$ induces necessary conditions on $g := g_0$ and $K := K_0$ to solve the Einstein equations:

$$Ric(h) - \frac{1}{2}R(h)h = T.$$

We now show how these necessary conditions are obtained. Let $\vec{n}$ be a unit normal vector to $M^n \times \{0\}$ in $\mathcal{M}^{n+1}$, represented by the index 0, and $(e_i)_i$ an orthonormal basis on M^n. The Einstein equations yield:

$$\begin{aligned} T_{i0} = \mathrm{Ric}_{i0} &= h^{\alpha\beta} Rm(h)_{\alpha i 0 \beta} \\ &= g^{kl} Rm(h)_{ki0l} \end{aligned}$$

The constraint equations

The structure $\mathcal{M}^{n+1} = (M^n \times \mathbb{R}, h)$ induces necessary conditions on $g := g_0$ and $K := K_0$ to solve the Einstein equations:

$$Ric(h) - \frac{1}{2}R(h)h = T.$$

We now show how these necessary conditions are obtained. Let $\vec{n}$ be a unit normal vector to $M^n \times \{0\}$ in $\mathcal{M}^{n+1}$, represented by the index 0, and $(e_i)_i$ an orthonormal basis on M^n. The Einstein equations yield:

$$\begin{aligned} T_{i0} = \mathrm{Ric}_{i0} &= h^{\alpha\beta} Rm(h)_{\alpha i 0 \beta} \\ &= g^{kl} Rm(h)_{ki0l} \\ &= g^{kl} \left(K_{il,k} - K_{kl,i} \right) \end{aligned}$$

The constraint equations

The structure $\mathcal{M}^{n+1} = (M^n \times \mathbb{R}, h)$ induces necessary conditions on $g := g_0$ and $K := K_0$ to solve the Einstein equations:

$$Ric(h) - \frac{1}{2} R(h) h = T.$$

We now show how these necessary conditions are obtained. Let $\vec{n}$ be a unit normal vector to $M^n \times \{0\}$ in $\mathcal{M}^{n+1}$, represented by the index 0, and $(e_i)_i$ an orthonormal basis on M^n. The Einstein equations yield:

$$\begin{aligned} T_{i0} = \mathrm{Ric}_{i0} &= h^{\alpha\beta} Rm(h)_{\alpha i 0 \beta} \\ &= g^{kl} Rm(h)_{ki0l} \\ &= g^{kl} \left(K_{il,k} - K_{kl,i} \right) \\ &= \mathrm{div}_g K - \nabla^g \mathrm{tr}_g K, \end{aligned}$$

The constraint equations

The structure $\mathcal{M}^{n+1} = (M^n \times \mathbb{R}, h)$ induces necessary conditions on $g := g_0$ and $K := K_0$ to solve the Einstein equations:

$$Ric(h) - \frac{1}{2}R(h)h = T.$$

We now show how these necessary conditions are obtained. Let $\vec{n}$ be a unit normal vector to $M^n \times \{0\}$ in $\mathcal{M}^{n+1}$, represented by the index 0, and $(e_i)_i$ an orthonormal basis on M^n. The Einstein equations yield:

$$\begin{aligned} T_{i0} = \mathrm{Ric}_{i0} &= h^{\alpha\beta} Rm(h)_{\alpha i 0 \beta} \\ &= g^{kl} Rm(h)_{ki0l} \\ &= g^{kl}\left(K_{il,k} - K_{kl,i}\right) \\ &= \mathrm{div}_g K - \nabla^g \mathrm{tr}_g K, \end{aligned}$$

where α, β run on space-time indices, k, l run on space indices, $g = h_{|M^n}$

The constraint equations

The structure $\mathcal{M}^{n+1} = (M^n \times \mathbb{R}, h)$ induces necessary conditions on $g := g_0$ and $K := K_0$ to solve the Einstein equations:

$$Ric(h) - \frac{1}{2}R(h)h = T.$$

We now show how these necessary conditions are obtained. Let $\vec{n}$ be a unit normal vector to $M^n \times \{0\}$ in $\mathcal{M}^{n+1}$, represented by the index 0, and $(e_i)_i$ an orthonormal basis on M^n. The Einstein equations yield:

$$\begin{aligned} T_{i0} = \mathrm{Ric}_{i0} &= h^{\alpha\beta} Rm(h)_{\alpha i 0 \beta} \\ &= g^{kl} Rm(h)_{ki0l} \\ &= g^{kl}\left(K_{il,k} - K_{kl,i}\right) \\ &= \mathrm{div}_g K - \nabla^g \mathrm{tr}_g K, \end{aligned}$$

where α, β run on space-time indices, k, l run on space indices, $g = h_{|M^n}$ and K is the second fundamental form of $M^n \times \{0\} \subset \mathcal{M}^n$.

The constraint equations

The structure $\mathcal{M}^{n+1} = (M^n \times \mathbb{R}, h)$ induces necessary conditions on $g := g_0$ and $K := K_0$ to solve the Einstein equations:

$$Ric(h) - \frac{1}{2} R(h) h = T.$$

We now show how these necessary conditions are obtained. Let $\vec{n}$ be a unit normal vector to $M^n \times \{0\}$ in $\mathcal{M}^{n+1}$, represented by the index 0, and $(e_i)_i$ an orthonormal basis on M^n. The Einstein equations yield:

$$\begin{aligned} T_{i0} = \mathrm{Ric}_{i0} &= h^{\alpha\beta} Rm(h)_{\alpha i 0 \beta} \\ &= g^{kl} Rm(h)_{ki0l} \\ &= g^{kl} \left(K_{il,k} - K_{kl,i} \right) \\ &= \mathrm{div}_g K - \nabla^g \mathrm{tr}_g K, \end{aligned}$$

where α, β run on space-time indices, k, l run on space indices, $g = h_{|M^n}$ and K is the second fundamental form of $M^n \times \{0\} \subset \mathcal{M}^n$. This is just the Codazzi equation.

The constraint equations II

An analogous computation using the Gauss equations shows that if $(M^n \times \mathbb{R}, h)$ solves the Einstein equations, then $g = h_{|M^n}$ and K satisfy the so-called constraint equations:

The constraint equations II

An analogous computation using the Gauss equations shows that if $(M^n \times \mathbb{R}, h)$ solves the Einstein equations, then $g = h_{|M^n}$ and K satisfy the so-called constraint equations:

$$\begin{cases} R(g) + (\mathrm{tr}_g K)^2 - |K|_g^2 = 2\rho, \\ \mathrm{div}_g K - \nabla(\mathrm{tr}_g K) = J, \end{cases} \qquad (\mathcal{C})$$

The constraint equations II

An analogous computation using the Gauss equations shows that if $(M^n \times \mathbb{R}, h)$ solves the Einstein equations, then $g = h_{|M^n}$ and K satisfy the so-called constraint equations:

$$\begin{cases} R(g) + (\mathrm{tr}_g K)^2 - |K|_g^2 = 2\rho, \\ \mathrm{div}_g K - \nabla(\mathrm{tr}_g K) = J, \end{cases} \tag{$\mathcal{C}$}$$

where we have let

$$\rho = T(\vec{n}, \vec{n}), \quad J = T(\vec{n}, \cdot).$$

The constraint equations II

An analogous computation using the Gauss equations shows that if $(M^n \times \mathbb{R}, h)$ solves the Einstein equations, then $g = h_{|M^n}$ and K satisfy the so-called constraint equations:

$$\begin{cases} R(g) + (\mathrm{tr}_g K)^2 - |K|_g^2 = 2\rho, \\ \mathrm{div}_g K - \nabla(\mathrm{tr}_g K) = J, \end{cases} \qquad (\mathcal{C})$$

where we have let

$$\rho = T(\vec{n}, \vec{n}), \quad J = T(\vec{n}, \cdot).$$

The constraint equations are a system of equations on M^n, whose unknowns are (g, K, ρ, J).

The constraint equations II

An analogous computation using the Gauss equations shows that if $(M^n \times \mathbb{R}, h)$ solves the Einstein equations, then $g = h_{|M^n}$ and K satisfy the so-called constraint equations:

$$\begin{cases} R(g) + (\mathrm{tr}_g K)^2 - |K|_g^2 = 2\rho, \\ \mathrm{div}_g K - \nabla(\mathrm{tr}_g K) = J, \end{cases} \qquad (\mathcal{C})$$

where we have let

$$\rho = T(\vec{n}, \vec{n}), \quad J = T(\vec{n}, \cdot).$$

The constraint equations are a system of equations on M^n, whose unknowns are (g, K, ρ, J). They are a necessary condition to the existence of solutions of the Einstein equations in $M^n \times \mathbb{R}$.

The constraint equations II

An analogous computation using the Gauss equations shows that if $(M^n \times \mathbb{R}, h)$ solves the Einstein equations, then $g = h_{|M^n}$ and K satisfy the so-called constraint equations:

$$\begin{cases} R(g) + (\mathrm{tr}_g K)^2 - |K|_g^2 = 2\rho, \\ \mathrm{div}_g K - \nabla(\mathrm{tr}_g K) = J, \end{cases} \tag{$\mathcal{C}$}$$

where we have let

$$\rho = T(\vec{n}, \vec{n}), \quad J = T(\vec{n}, \cdot).$$

The constraint equations are a system of equations on M^n, whose unknowns are (g, K, ρ, J). They are a necessary condition to the existence of solutions of the Einstein equations in $M^n \times \mathbb{R}$. A celebrated result of Choquet-Bruhat ('52) and Choquet-Bruhat-Geroch ('69) shows that they also are a sufficient condition.

The constraint equations II

An analogous computation using the Gauss equations shows that if $(M^n \times \mathbb{R}, h)$ solves the Einstein equations, then $g = h_{|M^n}$ and K satisfy the so-called constraint equations:

$$\begin{cases} R(g) + (\mathrm{tr}_g K)^2 - |K|_g^2 = 2\rho, \\ \mathrm{div}_g K - \nabla(\mathrm{tr}_g K) = J, \end{cases} \qquad (\mathcal{C})$$

where we have let

$$\rho = T(\vec{n}, \vec{n}), \quad J = T(\vec{n}, \cdot).$$

The constraint equations are a system of equations on M^n, whose unknowns are (g, K, ρ, J). They are a necessary condition to the existence of solutions of the Einstein equations in $M^n \times \mathbb{R}$. A celebrated result of Choquet-Bruhat ('52) and Choquet-Bruhat-Geroch ('69) shows that they also are a sufficient condition.

Solving $(\mathcal{C})$ therefore determines the admissible initial-data sets for the Einstein equations:

The constraint equations II

An analogous computation using the Gauss equations shows that if $(M^n \times \mathbb{R}, h)$ solves the Einstein equations, then $g = h_{|M^n}$ and K satisfy the so-called constraint equations:

$$\begin{cases} R(g) + (\mathrm{tr}_g K)^2 - |K|_g^2 = 2\rho, \\ \mathrm{div}_g K - \nabla(\mathrm{tr}_g K) = J, \end{cases} \qquad (\mathcal{C})$$

where we have let

$$\rho = T(\vec{n}, \vec{n}), \quad J = T(\vec{n}, \cdot).$$

The constraint equations are a system of equations on M^n, whose unknowns are (g, K, ρ, J). They are a necessary condition to the existence of solutions of the Einstein equations in $M^n \times \mathbb{R}$. A celebrated result of Choquet-Bruhat ('52) and Choquet-Bruhat-Geroch ('69) shows that they also are a sufficient condition.

Solving $(\mathcal{C})$ therefore determines the admissible initial-data sets for the Einstein equations: we'll focus on its resolution and on the properties of **some** of its solutions in the rest of this talk.

Our approach: parameterizing the initial-data sets

Let (M^n, g) be a Riemannian manifold and consider in M^n:

$$\begin{cases} R(g) + (\mathrm{tr}_g K)^2 - |K|_g^2 = 2\rho, \\ \mathrm{div}_g K - \nabla(\mathrm{tr}_g K) = J, \end{cases} \qquad (\mathcal{C})$$

Our approach: parameterizing the initial-data sets

Let (M^n, g) be a Riemannian manifold and consider in M^n:

$$\begin{cases} R(g) + (\mathrm{tr}_g K)^2 - |K|_g^2 = 2\rho, \\ \mathrm{div}_g K - \nabla(\mathrm{tr}_g K) = J, \end{cases} \qquad (\mathcal{C})$$

The main feature of the constraint equations is their underdetermination: they have $\geq n(n+1)$ unknowns for $n+1$ equations.

Our approach: parameterizing the initial-data sets

Let (M^n, g) be a Riemannian manifold and consider in M^n:

$$\begin{cases} R(g) + (\mathrm{tr}_g K)^2 - |K|_g^2 = 2\rho, \\ \mathrm{div}_g K - \nabla(\mathrm{tr}_g K) = J, \end{cases} \qquad (\mathcal{C})$$

The main feature of the constraint equations is their underdetermination: they have $\geq n(n+1)$ unknowns for $n+1$ equations.

Today we will approach the constraint equations via the so-called conformal method

Our approach: parameterizing the initial-data sets

Let (M^n, g) be a Riemannian manifold and consider in M^n:

$$\begin{cases} R(g) + (\mathrm{tr}_g K)^2 - |K|_g^2 = 2\rho, \\ \mathrm{div}_g K - \nabla(\mathrm{tr}_g K) = J, \end{cases} \qquad (\mathcal{C})$$

The main feature of the constraint equations is their underdetermination: they have $\geq n(n+1)$ unknowns for $n+1$ equations.

Today we will approach the constraint equations via the so-called conformal method which aims at making the system determined

Our approach: parameterizing the initial-data sets

Let (M^n, g) be a Riemannian manifold and consider in M^n:

$$\begin{cases} R(g) + (\mathrm{tr}_g K)^2 - |K|_g^2 = 2\rho, \\ \mathrm{div}_g K - \nabla(\mathrm{tr}_g K) = J, \end{cases} \qquad (\mathcal{C})$$

The main feature of the constraint equations is their underdetermination: they have $\geq n(n+1)$ unknowns for $n+1$ equations.

Today we will approach the constraint equations via the so-called conformal method which aims at making the system determined by obtaining an explicit parameterization of initial-data sets in terms of an arbitrary subset of data.

Our approach: parameterizing the initial-data sets

Let (M^n, g) be a Riemannian manifold and consider in M^n:

$$\begin{cases} R(g) + (\mathrm{tr}_g K)^2 - |K|_g^2 = 2\rho, \\ \mathrm{div}_g K - \nabla(\mathrm{tr}_g K) = J, \end{cases} \qquad (\mathcal{C})$$

The main feature of the constraint equations is their underdetermination: they have $\geq n(n+1)$ unknowns for $n+1$ equations.

Today we will approach the constraint equations via the so-called conformal method which aims at making the system determined by obtaining an explicit parameterization of initial-data sets in terms of an arbitrary subset of data.

We won't dwell on other techniques, e.g. glueing methods, which proved extremely useful to construct solutions of $(\mathcal{C})$ in a variety of geometric settings (see e.g. Chruściel-Isenberg-Pollack '05, Corvino-Schoen '06, Carlotto-Schoen '15, Chruściel-Delay '18, etc...).

The conformal method

The conformal method was initiated by Lichnerowicz ('48) in the maximal case $\tau \equiv 0$

The conformal method

The conformal method was initiated by Lichnerowicz ('48) in the maximal case $\tau \equiv 0$ an later developed by Choquet-Bruhat and York ('74).

The conformal method

The conformal method was initiated by Lichnerowicz ('48) in the maximal case $\tau \equiv 0$ an later developed by Choquet-Bruhat and York ('74).

To make the system determined, we reduce the number of unknowns and arbitrarily specify some of them

The conformal method

The conformal method was initiated by Lichnerowicz ('48) in the maximal case $\tau \equiv 0$ an later developed by Choquet-Bruhat and York ('74).

To make the system determined, we reduce the number of unknowns and arbitrarily specify some of them which are then considered as a data of the problem.

The conformal method

The conformal method was initiated by Lichnerowicz ('48) in the maximal case $\tau \equiv 0$ an later developed by Choquet-Bruhat and York ('74).

To make the system determined, we reduce the number of unknowns and arbitrarily specify some of them which are then considered as a data of the problem. One looks for initial data (g, K) as:

The conformal method

The conformal method was initiated by Lichnerowicz ('48) in the maximal case $\tau \equiv 0$ an later developed by Choquet-Bruhat and York ('74).

To make the system determined, we reduce the number of unknowns and arbitrarily specify some of them which are then considered as a data of the problem. One looks for initial data (g, K) as:

- $g = u^{\frac{4}{n-2}} g_0$, for some unknown $u > 0$,

The conformal method

The conformal method was initiated by Lichnerowicz ('48) in the maximal case $\tau \equiv 0$ an later developed by Choquet-Bruhat and York ('74).

To make the system determined, we reduce the number of unknowns and arbitrarily specify some of them which are then considered as a data of the problem. One looks for initial data (g, K) as:

- $g = u^{\frac{4}{n-2}} g_0$, for some unknown $u > 0$, where g_0 is some fixed Riemannian metric in M^n.

The conformal method

The conformal method was initiated by Lichnerowicz ('48) in the maximal case $\tau \equiv 0$ an later developed by Choquet-Bruhat and York ('74).

To make the system determined, we reduce the number of unknowns and arbitrarily specify some of them which are then considered as a data of the problem. One looks for initial data (g, K) as:

- $g = u^{\frac{4}{n-2}} g_0$, for some unknown $u > 0$, where g_0 is some fixed Riemannian metric in M^n.
- $$K = \frac{\tau}{n} u^{\frac{4}{n-2}} g_0 +$$

The conformal method

The conformal method was initiated by Lichnerowicz ('48) in the maximal case $\tau \equiv 0$ an later developed by Choquet-Bruhat and York ('74).

To make the system determined, we reduce the number of unknowns and arbitrarily specify some of them which are then considered as a data of the problem. One looks for initial data (g, K) as:

- $g = u^{\frac{4}{n-2}} g_0$, for some unknown $u > 0$, where g_0 is some fixed Riemannian metric in M^n.
- $$K = \frac{\tau}{n} u^{\frac{4}{n-2}} g_0 + u^{-2} \left(\sigma + \mathcal{L}_{g_0} W \right),$$

The conformal method

The conformal method was initiated by Lichnerowicz ('48) in the maximal case $\tau \equiv 0$ an later developed by Choquet-Bruhat and York ('74).

To make the system determined, we reduce the number of unknowns and arbitrarily specify some of them which are then considered as a data of the problem. One looks for initial data (g, K) as:

- $g = u^{\frac{4}{n-2}} g_0$, for some unknown $u > 0$, where g_0 is some fixed Riemannian metric in M^n.
- $$K = \frac{\tau}{n} u^{\frac{4}{n-2}} g_0 + u^{-2} \left(\sigma + \mathcal{L}_{g_0} W \right),$$

  where $W \in \bigwedge^1 (T^* M)$ is unknown, $\mathrm{div}_{g_0} \sigma = 0$, $\mathrm{tr}_{g_0} \sigma = 0$ is given and

  $$\mathcal{L}_{g_0} W_{ij} = \nabla_i W_j + \nabla_j W_i - \frac{2}{n} \mathrm{div}_{g_0} W (g_0)_{ij}.$$

The conformal method

The conformal method was initiated by Lichnerowicz ('48) in the maximal case $\tau \equiv 0$ an later developed by Choquet-Bruhat and York ('74).

To make the system determined, we reduce the number of unknowns and arbitrarily specify some of them which are then considered as a data of the problem. One looks for initial data (g, K) as:

- $g = u^{\frac{4}{n-2}} g_0$, for some unknown $u > 0$, where g_0 is some fixed Riemannian metric in M^n.
- $$K = \frac{\tau}{n} u^{\frac{4}{n-2}} g_0 + u^{-2} \left(\sigma + \mathcal{L}_{g_0} W \right),$$

  where $W \in \bigwedge^1(T^*M)$ is unknown, $\mathrm{div}_{g_0}\sigma = 0$, $\mathrm{tr}_{g_0}\sigma = 0$ is given and

  $$\mathcal{L}_{g_0} W_{ij} = \nabla_i W_j + \nabla_j W_i - \frac{2}{n} \mathrm{div}_{g_0} W (g_0)_{ij}.$$

In other words, we specified the conformal class of g,

The conformal method

The conformal method was initiated by Lichnerowicz ('48) in the maximal case $\tau \equiv 0$ an later developed by Choquet-Bruhat and York ('74).

To make the system determined, we reduce the number of unknowns and arbitrarily specify some of them which are then considered as a data of the problem. One looks for initial data (g, K) as:

- $g = u^{\frac{4}{n-2}} g_0$, for some unknown $u > 0$, where g_0 is some fixed Riemannian metric in M^n.
- $$K = \frac{\tau}{n} u^{\frac{4}{n-2}} g_0 + u^{-2} \left(\sigma + \mathcal{L}_{g_0} W \right),$$

  where $W \in \bigwedge^1 (T^* M)$ is unknown, $\mathrm{div}_{g_0} \sigma = 0$, $\mathrm{tr}_{g_0} \sigma = 0$ is given and

  $$\mathcal{L}_{g_0} W_{ij} = \nabla_i W_j + \nabla_j W_i - \frac{2}{n} \mathrm{div}_{g_0} W (g_0)_{ij}.$$

In other words, we specified the conformal class of g, the trace τ of K

The conformal method

The conformal method was initiated by Lichnerowicz ('48) in the maximal case $\tau \equiv 0$ an later developed by Choquet-Bruhat and York ('74).

To make the system determined, we reduce the number of unknowns and arbitrarily specify some of them which are then considered as a data of the problem. One looks for initial data (g, K) as:

- $g = u^{\frac{4}{n-2}} g_0$, for some unknown $u > 0$, where g_0 is some fixed Riemannian metric in M^n.
- $$K = \frac{\tau}{n} u^{\frac{4}{n-2}} g_0 + u^{-2} \left(\sigma + \mathcal{L}_{g_0} W \right),$$

  where $W \in \bigwedge^1(T^*M)$ is unknown, $\mathrm{div}_{g_0} \sigma = 0$, $\mathrm{tr}_{g_0} \sigma = 0$ is given and

  $$\mathcal{L}_{g_0} W_{ij} = \nabla_i W_j + \nabla_j W_i - \frac{2}{n} \mathrm{div}_{g_0} W (g_0)_{ij}.$$

In other words, we specified the conformal class of g, the trace τ of K and its divergence-free and trace-free part (up to u).

The scalar constraint equation I

Consider for instance the scalar constraint equation:

$$R(g) + (\mathrm{tr}_g K)^2 - |K|_g^2 = 2\rho.$$

The scalar constraint equation I

Consider for instance the scalar constraint equation:

$$R(g) + (\mathrm{tr}_g K)^2 - |K|_g^2 = 2\rho.$$

The scalar curvature of $g = u^{\frac{4}{n-2}} g_0$ satisfies the conformal invariance relation:

$$R(u^{\frac{4}{n-2}} g_0) = u^{-\frac{n+2}{n-2}} \left(\frac{4(n-1)}{n-2} \triangle_{g_0} u + R(g_0) u \right),$$

The scalar constraint equation I

Consider for instance the scalar constraint equation:

$$R(g) + (\mathrm{tr}_g K)^2 - |K|_g^2 = 2\rho.$$

The scalar curvature of $g = u^{\frac{4}{n-2}} g_0$ satisfies the conformal invariance relation:

$$R(u^{\frac{4}{n-2}} g_0) = u^{-\frac{n+2}{n-2}} \left(\frac{4(n-1)}{n-2} \triangle_{g_0} u + R(g_0) u \right),$$

so that the scalar constraint equation becomes:

$$u^{-\frac{n+2}{n-2}} \left(\frac{4(n-1)}{n-2} \triangle_{g_0} u + R(g_0) u \right) =$$

The scalar constraint equation I

Consider for instance the scalar constraint equation:

$$R(g) + (\mathrm{tr}_g K)^2 - |K|_g^2 = 2\rho.$$

The scalar curvature of $g = u^{\frac{4}{n-2}} g_0$ satisfies the conformal invariance relation:

$$R(u^{\frac{4}{n-2}} g_0) = u^{-\frac{n+2}{n-2}} \left(\frac{4(n-1)}{n-2} \triangle_{g_0} u + R(g_0) u \right),$$

so that the scalar constraint equation becomes:

$$u^{-\frac{n+2}{n-2}} \left(\frac{4(n-1)}{n-2} \triangle_{g_0} u + R(g_0) u \right) = 2\rho - \tau^2 + u^{-\frac{8}{n-2}} |K|_{g_0}^2 .$$

The scalar constraint equation I

Consider for instance the scalar constraint equation:

$$R(g) + (\mathrm{tr}_g K)^2 - |K|_g^2 = 2\rho.$$

The scalar curvature of $g = u^{\frac{4}{n-2}} g_0$ satisfies the conformal invariance relation:

$$R(u^{\frac{4}{n-2}} g_0) = u^{-\frac{n+2}{n-2}} \left(\frac{4(n-1)}{n-2} \triangle_{g_0} u + R(g_0) u \right),$$

so that the scalar constraint equation becomes:

$$u^{-\frac{n+2}{n-2}} \left(\frac{4(n-1)}{n-2} \triangle_{g_0} u + R(g_0) u \right) = 2\rho - \tau^2 + u^{-\frac{8}{n-2}} |K|_{g_0}^2 .$$

Using the splitting

$$K = \frac{\tau}{n} u^{\frac{4}{n-2}} g_0 + u^{-2} (\sigma + \mathcal{L}_{g_0} W)$$

we get that u and W satisfy:

The scalar constraint equation I

Consider for instance the scalar constraint equation:

$$R(g) + (\mathrm{tr}_g K)^2 - |K|_g^2 = 2\rho.$$

The scalar curvature of $g = u^{\frac{4}{n-2}} g_0$ satisfies the conformal invariance relation:

$$R(u^{\frac{4}{n-2}} g_0) = u^{-\frac{n+2}{n-2}} \left(\frac{4(n-1)}{n-2} \triangle_{g_0} u + R(g_0) u \right),$$

so that the scalar constraint equation becomes:

$$u^{-\frac{n+2}{n-2}} \left(\frac{4(n-1)}{n-2} \triangle_{g_0} u + R(g_0) u \right) = 2\rho - \tau^2 + u^{-\frac{8}{n-2}} |K|_{g_0}^2 .$$

Using the splitting

$$K = \frac{\tau}{n} u^{\frac{4}{n-2}} g_0 + u^{-2} (\sigma + \mathcal{L}_{g_0} W)$$

we get that u and W satisfy:

$$\frac{4(n-1)}{n-2} \triangle_{g_0} u + R(g_0) u =$$

The scalar constraint equation I

Consider for instance the scalar constraint equation:

$$R(g) + (\mathrm{tr}_g K)^2 - |K|_g^2 = 2\rho.$$

The scalar curvature of $g = u^{\frac{4}{n-2}} g_0$ satisfies the conformal invariance relation:

$$R(u^{\frac{4}{n-2}} g_0) = u^{-\frac{n+2}{n-2}} \left(\frac{4(n-1)}{n-2} \triangle_{g_0} u + R(g_0) u \right),$$

so that the scalar constraint equation becomes:

$$u^{-\frac{n+2}{n-2}} \left(\frac{4(n-1)}{n-2} \triangle_{g_0} u + R(g_0) u \right) = 2\rho - \tau^2 + u^{-\frac{8}{n-2}} |K|_{g_0}^2 .$$

Using the splitting

$$K = \frac{\tau}{n} u^{\frac{4}{n-2}} g_0 + u^{-2} (\sigma + \mathcal{L}_{g_0} W)$$

we get that u and W satisfy:

$$\frac{4(n-1)}{n-2} \triangle_{g_0} u + R(g_0) u = \left(2\rho - \frac{n-1}{n} \tau^2 \right) u^{\frac{n+2}{n-2}} + |\sigma + \mathcal{L}_{g_0} W|_{g_0}^2 \, u^{-\frac{3n-2}{n-2}}.$$

The scalar constraint equation II

In the framework of a scalar-field theory ρ and J depend on $\Psi_{|M^n}$ and $\partial_t \Psi_{|M^n}$.

The scalar constraint equation II

In the framework of a scalar-field theory ρ and J depend on $\Psi_{|M^n}$ and $\partial_t \Psi_{|M^n}$. Precisely:

$$2\rho = 2V(\Psi_{|M^n}) + (\partial_t \Psi_{|M^n})^2 + \left|\nabla \Psi_{|M^n}\right|_g^2 .$$

The scalar constraint equation II

In the framework of a scalar-field theory ρ and J depend on $\Psi_{|M^n}$ and $\partial_t \Psi_{|M^n}$. Precisely:

$$2\rho = 2V(\Psi_{|M^n}) + (\partial_t \Psi_{|M^n})^2 + \left|\nabla \Psi_{|M^n}\right|_g^2 .$$

For convenience, we scale these two quantities too and let:

$$\psi = \Psi_{|M^n}$$

The scalar constraint equation II

In the framework of a scalar-field theory ρ and J depend on $\Psi_{|M^n}$ and $\partial_t \Psi_{|M^n}$. Precisely:

$$2\rho = 2V(\Psi_{|M^n}) + (\partial_t \Psi_{|M^n})^2 + \left|\nabla \Psi_{|M^n}\right|_g^2 .$$

For convenience, we scale these two quantities too and let:

$$\psi = \Psi_{|M^n} \quad \text{and} \quad \pi = u^{\frac{2n}{n-2}} \partial_t \Psi_{|M^n}.$$

The scalar constraint equation II

In the framework of a scalar-field theory ρ and J depend on $\Psi_{|M^n}$ and $\partial_t \Psi_{|M^n}$.
Precisely:

$$2\rho = 2V(\Psi_{|M^n}) + (\partial_t \Psi_{|M^n})^2 + \left|\nabla \Psi_{|M^n}\right|_g^2 .$$

For convenience, we scale these two quantities too and let:

$$\psi = \Psi_{|M^n} \quad \text{and} \quad \pi = u^{\frac{2n}{n-2}} \partial_t \Psi_{|M^n}.$$

Therefore

$$2\rho = 2V(\psi) + u^{-\frac{4n}{n-2}} \pi^2 + u^{-\frac{4}{n-2}} |\nabla \psi|_{g_0}^2,$$

The scalar constraint equation II

In the framework of a scalar-field theory ρ and J depend on $\Psi_{|M^n}$ and $\partial_t \Psi_{|M^n}$. Precisely:

$$2\rho = 2V(\Psi_{|M^n}) + (\partial_t \Psi_{|M^n})^2 + \left|\nabla \Psi_{|M^n}\right|_g^2 .$$

For convenience, we scale these two quantities too and let:

$$\psi = \Psi_{|M^n} \quad \text{and} \quad \pi = u^{\frac{2n}{n-2}} \partial_t \Psi_{|M^n}.$$

Therefore

$$2\rho = 2V(\psi) + u^{-\frac{4n}{n-2}} \pi^2 + u^{-\frac{4}{n-2}} |\nabla \psi|_{g_0}^2,$$

so that the scalar constraint equation becomes:

$$\frac{4(n-1)}{n-2} \triangle_{g_0} u + (R(g_0) - |\nabla \psi|_g^2) u = \left(2V(\psi) - \frac{n-1}{n}\tau^2\right) u^{\frac{n+2}{n-2}} + \frac{\pi^2 + |\sigma + \mathcal{L}_{g_0} W|_{g_0}^2}{u^{\frac{3n-2}{n-2}}}.$$

The Einstein-Lichnerowicz system

Transforming in the same way the vector equation shows that an initial-data set

$$(g, K, \rho, J) = \left(u^{\frac{4}{n-2}} g_0, \frac{\tau}{n} u^{\frac{4}{n-2}} g_0 + u^{-2} \left(\sigma + \mathcal{L}_{g_0} W\right), \psi, u^{-\frac{2n}{n-2}} \pi\right) \tag{1.1}$$

solves the constraint equations in M^n

The Einstein-Lichnerowicz system

Transforming in the same way the vector equation shows that an initial-data set

$$(g, K, \rho, J) = \left(u^{\frac{4}{n-2}} g_0, \frac{\tau}{n} u^{\frac{4}{n-2}} g_0 + u^{-2}\left(\sigma + \mathcal{L}_{g_0} W\right), \psi, u^{-\frac{2n}{n-2}} \pi\right) \tag{1.1}$$

solves the constraint equations in M^n if and only if (u, W) satisfy the so-called Einstein-Lichnerowicz system in M^n:

The Einstein-Lichnerowicz system

Transforming in the same way the vector equation shows that an initial-data set

$$(g, K, \rho, J) = \left(u^{\frac{4}{n-2}} g_0, \frac{\tau}{n} u^{\frac{4}{n-2}} g_0 + u^{-2} \left(\sigma + \mathcal{L}_{g_0} W\right), \psi, u^{-\frac{2n}{n-2}} \pi\right) \qquad (1.1)$$

solves the constraint equations in M^n if and only if (u, W) satisfy the so-called Einstein-Lichnerowicz system in M^n:

$$\begin{cases} \frac{4(n-1)}{n-2} \triangle_{g_0} u + h(g_0, \psi) u = f(\tau, \psi, V) u^{2^*-1} + \frac{\pi^2 + |\sigma + \mathcal{L}_{g_0} W|^2_{g_0}}{u^{2^*+1}} , \\ \end{cases}$$

The Einstein-Lichnerowicz system

Transforming in the same way the vector equation shows that an initial-data set

$$(g, K, \rho, J) = \left(u^{\frac{4}{n-2}} g_0, \frac{\tau}{n} u^{\frac{4}{n-2}} g_0 + u^{-2}\left(\sigma + \mathcal{L}_{g_0} W\right), \psi, u^{-\frac{2n}{n-2}} \pi\right) \qquad (1.1)$$

solves the constraint equations in M^n if and only if (u, W) satisfy the so-called Einstein-Lichnerowicz system in M^n:

$$\begin{cases} \frac{4(n-1)}{n-2} \triangle_{g_0} u + h(g_0, \psi) u = f(\tau, \psi, V) u^{2^*-1} + \frac{\pi^2 + |\sigma + \mathcal{L}_{g_0} W|^2_{g_0}}{u^{2^*+1}}, \\ \overrightarrow{\triangle}_{g_0} W = -\frac{n-1}{n} u^{2^*} \nabla \tau - \pi \nabla \psi, \end{cases} \qquad (1.2)$$

The Einstein-Lichnerowicz system

Transforming in the same way the vector equation shows that an initial-data set

$$(g, K, \rho, J) = \left(u^{\frac{4}{n-2}} g_0, \frac{\tau}{n} u^{\frac{4}{n-2}} g_0 + u^{-2} \left(\sigma + \mathcal{L}_{g_0} W\right), \psi, u^{-\frac{2n}{n-2}} \pi\right) \tag{1.1}$$

solves the constraint equations in M^n if and only if (u, W) satisfy the so-called Einstein-Lichnerowicz system in M^n:

$$\begin{cases} \frac{4(n-1)}{n-2} \triangle_{g_0} u + h(g_0, \psi) u = f(\tau, \psi, V) u^{2^*-1} + \frac{\pi^2 + |\sigma + \mathcal{L}_{g_0} W|^2_{g_0}}{u^{2^*+1}}, \\ \overrightarrow{\triangle}_{g_0} W = -\frac{n-1}{n} u^{2^*} \nabla \tau - \pi \nabla \psi, \end{cases} \tag{1.2}$$

where in (**??**) we have let $2^* = \frac{2n}{n-2}$,

The Einstein-Lichnerowicz system

Transforming in the same way the vector equation shows that an initial-data set

$$(g, K, \rho, J) = \left(u^{\frac{4}{n-2}} g_0, \frac{\tau}{n} u^{\frac{4}{n-2}} g_0 + u^{-2}\left(\sigma + \mathcal{L}_{g_0} W\right), \psi, u^{-\frac{2n}{n-2}} \pi\right) \qquad (1.1)$$

solves the constraint equations in M^n if and only if (u, W) satisfy the so-called Einstein-Lichnerowicz system in M^n:

$$\begin{cases} \frac{4(n-1)}{n-2} \triangle_{g_0} u + h(g_0, \psi) u = f(\tau, \psi, V) u^{2^*-1} + \frac{\pi^2 + |\sigma + \mathcal{L}_{g_0} W|^2_{g_0}}{u^{2^*+1}} , \\ \overrightarrow{\triangle}_{g_0} W = -\frac{n-1}{n} u^{2^*} \nabla \tau - \pi \nabla \psi, \end{cases} \qquad (1.2)$$

where in (**??**) we have let $2^* = \frac{2n}{n-2}$, $\triangle_{g_0} = -div_{g_0}(\nabla \cdot)$,

The Einstein-Lichnerowicz system

Transforming in the same way the vector equation shows that an initial-data set

$$(g, K, \rho, J) = \left(u^{\frac{4}{n-2}} g_0, \frac{\tau}{n} u^{\frac{4}{n-2}} g_0 + u^{-2}\left(\sigma + \mathcal{L}_{g_0} W\right), \psi, u^{-\frac{2n}{n-2}} \pi\right) \tag{1.1}$$

solves the constraint equations in M^n if and only if (u, W) satisfy the so-called Einstein-Lichnerowicz system in M^n:

$$\begin{cases} \frac{4(n-1)}{n-2} \triangle_{g_0} u + h(g_0, \psi) u = f(\tau, \psi, V) u^{2^*-1} + \frac{\pi^2 + |\sigma + \mathcal{L}_{g_0} W|^2_{g_0}}{u^{2^*+1}} , \\ \overrightarrow{\triangle}_{g_0} W = -\frac{n-1}{n} u^{2^*} \nabla \tau - \pi \nabla \psi, \end{cases} \tag{1.2}$$

where in (**??**) we have let $2^* = \frac{2n}{n-2}$, $\triangle_{g_0} = -div_{g_0}(\nabla \cdot)$, $\overrightarrow{\triangle}_{g_0} W = -div_{g_0}\left(\mathcal{L}_{g_0} W\right)$,

The Einstein-Lichnerowicz system

Transforming in the same way the vector equation shows that an initial-data set

$$(g, K, \rho, J) = \left(u^{\frac{4}{n-2}} g_0, \frac{\tau}{n} u^{\frac{4}{n-2}} g_0 + u^{-2}\left(\sigma + \mathcal{L}_{g_0} W\right), \psi, u^{-\frac{2n}{n-2}} \pi\right) \qquad (1.1)$$

solves the constraint equations in M^n if and only if (u, W) satisfy the so-called Einstein-Lichnerowicz system in M^n:

$$\begin{cases} \frac{4(n-1)}{n-2} \triangle_{g_0} u + h(g_0, \psi) u = f(\tau, \psi, V) u^{2^*-1} + \frac{\pi^2 + |\sigma + \mathcal{L}_{g_0} W|^2_{g_0}}{u^{2^*+1}} , \\ \overrightarrow{\triangle}_{g_0} W = -\frac{n-1}{n} u^{2^*} \nabla \tau - \pi \nabla \psi, \end{cases} \qquad (1.2)$$

where in (**??**) we have let $2^* = \frac{2n}{n-2}$, $\triangle_{g_0} = -div_{g_0}(\nabla \cdot)$, $\overrightarrow{\triangle}_{g_0} W = -div_{g_0}(\mathcal{L}_{g_0} W)$, and where the coefficient functions h and f depend on the physics data (τ, ψ, π, V) that we arbitrarily chose in the conformal method:

The Einstein-Lichnerowicz system

Transforming in the same way the vector equation shows that an initial-data set

$$(g, K, \rho, J) = \left(u^{\frac{4}{n-2}} g_0, \frac{\tau}{n} u^{\frac{4}{n-2}} g_0 + u^{-2}\left(\sigma + \mathcal{L}_{g_0} W\right), \psi, u^{-\frac{2n}{n-2}} \pi\right) \quad (1.1)$$

solves the constraint equations in M^n if and only if (u, W) satisfy the so-called Einstein-Lichnerowicz system in M^n:

$$\begin{cases} \frac{4(n-1)}{n-2} \triangle_{g_0} u + h(g_0, \psi) u = f(\tau, \psi, V) u^{2^*-1} + \frac{\pi^2 + |\sigma + \mathcal{L}_{g_0} W|^2_{g_0}}{u^{2^*+1}}, \\ \overrightarrow{\triangle}_{g_0} W = -\frac{n-1}{n} u^{2^*} \nabla \tau - \pi \nabla \psi, \end{cases} \quad (1.2)$$

where in (**??**) we have let $2^* = \frac{2n}{n-2}$, $\triangle_{g_0} = -div_{g_0}(\nabla \cdot)$, $\overrightarrow{\triangle}_{g_0} W = -div_{g_0}(\mathcal{L}_{g_0} W)$, and where the coefficient functions h and f depend on the physics data (τ, ψ, π, V) that we arbitrarily chose in the conformal method:

$$h(g_0, \psi) = R(g_0) - |\nabla \psi|^2_g$$

The Einstein-Lichnerowicz system

Transforming in the same way the vector equation shows that an initial-data set

$$(g, K, \rho, J) = \left(u^{\frac{4}{n-2}} g_0, \frac{\tau}{n} u^{\frac{4}{n-2}} g_0 + u^{-2}\left(\sigma + \mathcal{L}_{g_0} W\right), \psi, u^{-\frac{2n}{n-2}} \pi\right) \tag{1.1}$$

solves the constraint equations in M^n if and only if (u, W) satisfy the so-called Einstein-Lichnerowicz system in M^n:

$$\begin{cases} \frac{4(n-1)}{n-2} \triangle_{g_0} u + h(g_0, \psi) u = f(\tau, \psi, V) u^{2^*-1} + \frac{\pi^2 + |\sigma + \mathcal{L}_{g_0} W|_{g_0}^2}{u^{2^*+1}} , \\ \overrightarrow{\triangle}_{g_0} W = -\frac{n-1}{n} u^{2^*} \nabla \tau - \pi \nabla \psi, \end{cases} \tag{1.2}$$

where in (**??**) we have let $2^* = \frac{2n}{n-2}$, $\triangle_{g_0} = -div_{g_0}(\nabla \cdot)$, $\overrightarrow{\triangle}_{g_0} W = -div_{g_0}(\mathcal{L}_{g_0} W)$, and where the coefficient functions h and f depend on the physics data (τ, ψ, π, V) that we arbitrarily chose in the conformal method:

$$h(g_0, \psi) = R(g_0) - |\nabla \psi|_g^2 \quad , \quad f(\tau, \psi, V) = 2V(\psi) - \frac{n-1}{n} \tau^2.$$

Contents

The setting I

The setting I

From now on, (M^n, g) will denote a compact Riemannian manifold without boundary.

The setting I

From now on, (M^n, g) will denote a compact Riemannian manifold without boundary. For simplicity we replace g_0 by g and look for solutions (u, W) in M, $u > 0$, of the Einstein-Lichnerowicz system:

The setting I

From now on, (M^n, g) will denote a compact Riemannian manifold without boundary. For simplicity we replace g_0 by g and look for solutions (u, W) in M, $u > 0$, of the Einstein-Lichnerowicz system:

$$\begin{cases} \frac{4(n-1)}{n-2}\triangle_g u + h(g,\psi)u = f(\tau,\psi,V)u^{2^*-1} + \frac{\pi^2 + |\sigma + \mathcal{L}_g W|_g^2}{u^{2^*+1}}, \\ \overrightarrow{\triangle}_g W = -\frac{n-1}{n}u^{2^*}\nabla\tau - \pi\nabla\psi, \end{cases} \qquad (S_{\mathcal{D}})$$

The setting I

From now on, (M^n, g) will denote a compact Riemannian manifold without boundary. For simplicity we replace g_0 by g and look for solutions (u, W) in M, $u > 0$, of the Einstein-Lichnerowicz system:

$$\begin{cases} \frac{4(n-1)}{n-2} \triangle_g u + h(g,\psi) u = f(\tau, \psi, V) u^{2^*-1} + \frac{\pi^2 + |\sigma + \mathcal{L}_g W|_g^2}{u^{2^*+1}} , \\ \overrightarrow{\triangle}_g W = -\frac{n-1}{n} u^{2^*} \nabla\tau - \pi\nabla\psi, \end{cases} \qquad (S_{\mathcal{D}})$$

with $h(g_0, \psi) = R(g_0) - |\nabla\psi|_g^2$ and:

The setting I

From now on, (M^n, g) will denote a compact Riemannian manifold without boundary. For simplicity we replace g_0 by g and look for solutions (u, W) in M, $u > 0$, of the Einstein-Lichnerowicz system:

$$\begin{cases} \frac{4(n-1)}{n-2} \triangle_g u + h(g, \psi) u = f(\tau, \psi, V) u^{2^*-1} + \frac{\pi^2 + |\sigma + \mathcal{L}_g W|_g^2}{u^{2^*+1}} , \\ \overrightarrow{\triangle}_g W = -\frac{n-1}{n} u^{2^*} \nabla \tau - \pi \nabla \psi, \end{cases} \qquad (S_{\mathcal{D}})$$

with $h(g_0, \psi) = R(g_0) - |\nabla \psi|_g^2$ and:

$$f(\tau, \psi, V) = 2V(\psi) - \frac{n-1}{n} \tau^2.$$

The setting I

From now on, (M^n, g) will denote a compact Riemannian manifold without boundary. For simplicity we replace g_0 by g and look for solutions (u, W) in M, $u > 0$, of the Einstein-Lichnerowicz system:

$$\begin{cases} \frac{4(n-1)}{n-2} \triangle_g u + h(g,\psi) u = f(\tau, \psi, V) u^{2^*-1} + \frac{\pi^2 + |\sigma + \mathcal{L}_g W|_g^2}{u^{2^*+1}} , \\ \overrightarrow{\triangle}_g W = -\frac{n-1}{n} u^{2^*} \nabla \tau - \pi \nabla \psi, \end{cases} \qquad (S_{\mathcal{D}})$$

with $h(g_0, \psi) = R(g_0) - |\nabla \psi|_g^2$ and:

$$f(\tau, \psi, V) = 2V(\psi) - \frac{n-1}{n} \tau^2.$$

The coefficients $(V, \psi, \pi, \tau, \sigma)$ are the part of the initial-data that was arbitrarily fixed during the conformal method.

The setting I

From now on, (M^n, g) will denote a compact Riemannian manifold without boundary. For simplicity we replace g_0 by g and look for solutions (u, W) in M, $u > 0$, of the Einstein-Lichnerowicz system:

$$\begin{cases} \frac{4(n-1)}{n-2} \triangle_g u + h(g,\psi)u = f(\tau,\psi,V)u^{2^*-1} + \frac{\pi^2 + |\sigma + \mathcal{L}_g W|_g^2}{u^{2^*+1}}, \\ \overrightarrow{\triangle}_g W = -\frac{n-1}{n} u^{2^*} \nabla\tau - \pi\nabla\psi, \end{cases} \qquad (S_{\mathcal{D}})$$

with $h(g_0, \psi) = R(g_0) - |\nabla\psi|_g^2$ and:

$$f(\tau, \psi, V) = 2V(\psi) - \frac{n-1}{n}\tau^2.$$

The coefficients $(V, \psi, \pi, \tau, \sigma)$ are the part of the initial-data that was arbitrarily fixed during the conformal method. We shall call it the physics data and label it $\mathcal{D} = (V, \psi, \pi, \tau, \sigma)$.

The setting I

From now on, (M^n, g) will denote a compact Riemannian manifold without boundary. For simplicity we replace g_0 by g and look for solutions (u, W) in M, $u > 0$, of the Einstein-Lichnerowicz system:

$$\begin{cases} \frac{4(n-1)}{n-2}\triangle_g u + h(g,\psi)u = f(\tau,\psi,V)u^{2^*-1} + \frac{\pi^2 + |\sigma + \mathcal{L}_g W|_g^2}{u^{2^*+1}}, \\ \overrightarrow{\triangle}_g W = -\frac{n-1}{n}u^{2^*}\nabla\tau - \pi\nabla\psi, \end{cases} \qquad (S_{\mathcal{D}})$$

with $h(g_0,\psi) = R(g_0) - |\nabla\psi|_g^2$ and:

$$f(\tau,\psi,V) = 2V(\psi) - \frac{n-1}{n}\tau^2.$$

The coefficients (V,ψ,π,τ,σ) are the part of the initial-data that was arbitrarily fixed during the conformal method. We shall call it the physics data and label it $\mathcal{D} = (V,\psi,\pi,\tau,\sigma)$. We will sometimes say that (**??**) is the Einstein-Lichnerowicz system with physics data $\mathcal{D}$.

The setting II: focusing and defocusing case

$$\begin{cases} \frac{4(n-1)}{n-2}\triangle_g u + h(g,\psi)u = f(\tau,\psi,V)u^{2^*-1} + \frac{\pi^2 + |\sigma + \mathcal{L}_g W|_g^2}{u^{2^*+1}} \ , \\ \overrightarrow{\triangle}_g W = -\frac{n-1}{n} u^{2^*} \nabla\tau - \pi\nabla\psi, \end{cases} \qquad (S_{\mathcal{D}})$$

The setting II: focusing and defocusing case

$$\begin{cases} \frac{4(n-1)}{n-2} \triangle_g u + h(g,\psi) u = f(\tau,\psi,V) u^{2^*-1} + \frac{\pi^2 + |\sigma + \mathcal{L}_g W|_g^2}{u^{2^*+1}} , \\ \overrightarrow{\triangle}_g W = -\frac{n-1}{n} u^{2^*} \nabla \tau - \pi \nabla \psi, \end{cases} \qquad (S_{\mathcal{D}})$$

Definition

We say that $S_{\mathcal{D}}$ is

The setting II: focusing and defocusing case

$$\begin{cases} \frac{4(n-1)}{n-2} \triangle_g u + h(g,\psi) u = f(\tau,\psi,V) u^{2^*-1} + \frac{\pi^2 + |\sigma + \mathcal{L}_g W|_g^2}{u^{2^*+1}} \,, \\ \overrightarrow{\triangle}_g W = -\frac{n-1}{n} u^{2^*} \nabla \tau - \pi \nabla \psi, \end{cases} \qquad (S_{\mathcal{D}})$$

Definition

We say that $S_{\mathcal{D}}$ is

- in the focusing case if $f(\tau,\psi,V) > 0$.

The setting II: focusing and defocusing case

$$\begin{cases} \frac{4(n-1)}{n-2}\triangle_g u + h(g,\psi)u = f(\tau,\psi,V)u^{2^*-1} + \frac{\pi^2 + |\sigma + \mathcal{L}_g W|_g^2}{u^{2^*+1}}, \\ \overrightarrow{\triangle}_g W = -\frac{n-1}{n}u^{2^*}\nabla\tau - \pi\nabla\psi, \end{cases} \qquad (S_{\mathcal{D}})$$

Definition

We say that $S_{\mathcal{D}}$ is

- in the focusing case if $f(\tau,\psi,V) > 0$.
- in the defocusing case if $f(\tau,\psi,V) \leq 0$.

The setting II: focusing and defocusing case

$$\begin{cases} \frac{4(n-1)}{n-2}\triangle_g u + h(g,\psi)u = f(\tau,\psi,V)u^{2^*-1} + \frac{\pi^2 + |\sigma + \mathcal{L}_g W|_g^2}{u^{2^*+1}}, \\ \overrightarrow{\triangle}_g W = -\frac{n-1}{n}u^{2^*}\nabla\tau - \pi\nabla\psi, \end{cases} \qquad (S_{\mathcal{D}})$$

Definition

We say that $S_{\mathcal{D}}$ is

- in the focusing case if $f(\tau,\psi,V) > 0$.
- in the defocusing case if $f(\tau,\psi,V) \le 0$.

Recall that:

$$f(\tau,\psi,V) = 2V(\psi) - \frac{n-1}{n}\tau^2.$$

The setting II: focusing and defocusing case

$$\begin{cases} \dfrac{4(n-1)}{n-2}\triangle_g u + h(g,\psi)u = f(\tau,\psi,V)u^{2^*-1} + \dfrac{\pi^2 + |\sigma + \mathcal{L}_g W|_g^2}{u^{2^*+1}}, \\ \overrightarrow{\triangle}_g W = -\dfrac{n-1}{n} u^{2^*}\nabla\tau - \pi\nabla\psi, \end{cases} \qquad (S_{\mathcal{D}})$$

Definition

We say that $S_{\mathcal{D}}$ is

- in the focusing case if $f(\tau,\psi,V) > 0$.
- in the defocusing case if $f(\tau,\psi,V) \leq 0$.

Recall that:

$$f(\tau,\psi,V) = 2V(\psi) - \frac{n-1}{n}\tau^2.$$

So the vacuum case $\Psi \equiv 0$ and $V \equiv 0$, is defocusing.

The setting II: focusing and defocusing case

$$\begin{cases} \frac{4(n-1)}{n-2}\triangle_g u + h(g,\psi)u = f(\tau,\psi,V)u^{2^*-1} + \frac{\pi^2 + |\sigma + \mathcal{L}_g W|_g^2}{u^{2^*+1}}, \\ \overrightarrow{\triangle}_g W = -\frac{n-1}{n}u^{2^*}\nabla\tau - \pi\nabla\psi, \end{cases} \qquad (S_{\mathcal{D}})$$

Definition

We say that $S_{\mathcal{D}}$ is

- in the focusing case if $f(\tau,\psi,V) > 0$.
- in the defocusing case if $f(\tau,\psi,V) \leq 0$.

Recall that:

$$f(\tau,\psi,V) = 2V(\psi) - \frac{n-1}{n}\tau^2.$$

So the vacuum case $\Psi \equiv 0$ and $V \equiv 0$, is defocusing. In general, the focusing case is a matter case.

The setting II: focusing and defocusing case

$$\begin{cases} \frac{4(n-1)}{n-2}\triangle_g u + h(g,\psi)u = f(\tau,\psi,V)u^{2^*-1} + \frac{\pi^2 + |\sigma + \mathcal{L}_g W|_g^2}{u^{2^*+1}}, \\ \overrightarrow{\triangle}_g W = -\frac{n-1}{n}u^{2^*}\nabla\tau - \pi\nabla\psi, \end{cases} \qquad (S_{\mathcal{D}})$$

Definition

We say that $S_{\mathcal{D}}$ is

- in the focusing case if $f(\tau,\psi,V) > 0$.
- in the defocusing case if $f(\tau,\psi,V) \le 0$.

Recall that:

$$f(\tau,\psi,V) = 2V(\psi) - \frac{n-1}{n}\tau^2.$$

So the vacuum case $\Psi \equiv 0$ and $V \equiv 0$, is defocusing. In general, the focusing case is a matter case.

These two regimes exhibit a very different behavior analytically speaking.

An illustration: existence theory in the decoupled case

Under the assumption $\nabla \tau \equiv 0$

An illustration: existence theory in the decoupled case

Under the assumption $\nabla\tau \equiv 0$ (the spacetime mean curvature is constant) the system becomes the EL equation:

$$\frac{4(n-1)}{n-2}\triangle_g u + h(g,\psi)u = f(\tau,\psi,V)u^{2^*-1} + \frac{\pi^2 + |\sigma|_g^2}{u^{2^*+1}}.$$

An illustration: existence theory in the decoupled case

Under the assumption $\nabla\tau \equiv 0$ (the spacetime mean curvature is constant) the system becomes the EL equation:

$$\frac{4(n-1)}{n-2}\triangle_g u + h(g,\psi)u = f(\tau,\psi,V)u^{2^*-1} + \frac{\pi^2 + |\sigma|_g^2}{u^{2^*+1}}.$$

Here we assume that $(\pi,\sigma) \neq (0,0)$.

An illustration: existence theory in the decoupled case

Under the assumption $\nabla \tau \equiv 0$ (the spacetime mean curvature is constant) the system becomes the EL equation:

$$\frac{4(n-1)}{n-2} \triangle_g u + h(g, \psi) u = f(\tau, \psi, V) u^{2^*-1} + \frac{\pi^2 + |\sigma|_g^2}{u^{2^*+1}}.$$

Here we assume that $(\pi, \sigma) \neq (0,0)$. The behavior of the set of solutions changes drastically according to the sign of f:

An illustration: existence theory in the decoupled case

Under the assumption $\nabla\tau \equiv 0$ (the spacetime mean curvature is constant) the system becomes the EL equation:

$$\frac{4(n-1)}{n-2}\triangle_g u + h(g,\psi)u = f(\tau,\psi,V)u^{2^*-1} + \frac{\pi^2 + |\sigma|_g^2}{u^{2^*+1}}.$$

Here we assume that $(\pi,\sigma) \neq (0,0)$. The behavior of the set of solutions changes drastically according to the sign of f:

- When $f(\tau,\psi,V) \leq 0$ (defocusing), solutions are classified by the sub- and super-solution method (Isenberg '95)

An illustration: existence theory in the decoupled case

Under the assumption $\nabla\tau \equiv 0$ (the spacetime mean curvature is constant) the system becomes the EL equation:

$$\frac{4(n-1)}{n-2}\triangle_g u + h(g,\psi)u = f(\tau,\psi,V)u^{2^*-1} + \frac{\pi^2 + |\sigma|_g^2}{u^{2^*+1}}.$$

Here we assume that $(\pi,\sigma) \neq (0,0)$. The behavior of the set of solutions changes drastically according to the sign of f:

- When $f(\tau,\psi,V) \leq 0$ (defocusing), solutions are classified by the sub- and super-solution method (Isenberg '95) and are unique when they exist, except in one extreme case.

An illustration: existence theory in the decoupled case

Under the assumption $\nabla\tau \equiv 0$ (the spacetime mean curvature is constant) the system becomes the EL equation:

$$\frac{4(n-1)}{n-2}\triangle_g u + h(g,\psi)u = f(\tau,\psi,V)u^{2^*-1} + \frac{\pi^2 + |\sigma|_g^2}{u^{2^*+1}}.$$

Here we assume that $(\pi,\sigma) \neq (0,0)$. The behavior of the set of solutions changes drastically according to the sign of f:

- When $f(\tau,\psi,V) \leq 0$ (defocusing), solutions are classified by the sub- and super-solution method (Isenberg '95) and are unique when they exist, except in one extreme case.
- When $f(\tau,\psi,V) > 0$ (focusing), the first existence result is in Hebey-Pacard-Pollack ('07).

An illustration: existence theory in the decoupled case

Under the assumption $\nabla \tau \equiv 0$ (the spacetime mean curvature is constant) the system becomes the EL equation:

$$\frac{4(n-1)}{n-2} \triangle_g u + h(g,\psi) u = f(\tau,\psi,V) u^{2^*-1} + \frac{\pi^2 + |\sigma|_g^2}{u^{2^*+1}}.$$

Here we assume that $(\pi, \sigma) \neq (0,0)$. The behavior of the set of solutions changes drastically according to the sign of f:

- When $f(\tau,\psi,V) \leq 0$ (defocusing), solutions are classified by the sub- and super-solution method (Isenberg '95) and are unique when they exist, except in one extreme case.
- When $f(\tau,\psi,V) > 0$ (focusing), the first existence result is in Hebey-Pacard-Pollack ('07). Solutions only exist for a limited set of coefficients,

An illustration: existence theory in the decoupled case

Under the assumption $\nabla\tau \equiv 0$ (the spacetime mean curvature is constant) the system becomes the EL equation:

$$\frac{4(n-1)}{n-2}\triangle_g u + h(g,\psi)u = f(\tau,\psi,V)u^{2^*-1} + \frac{\pi^2 + |\sigma|_g^2}{u^{2^*+1}}.$$

Here we assume that $(\pi,\sigma) \neq (0,0)$. The behavior of the set of solutions changes drastically according to the sign of f:

- When $f(\tau,\psi,V) \leq 0$ (defocusing), solutions are classified by the sub- and super-solution method (Isenberg '95) and are unique when they exist, except in one extreme case.
- When $f(\tau,\psi,V) > 0$ (focusing), the first existence result is in Hebey-Pacard-Pollack ('07). Solutions only exist for a limited set of coefficients, and there are generically two solutions (P., '14).

An illustration: existence theory in the decoupled case

Under the assumption $\nabla\tau \equiv 0$ (the spacetime mean curvature is constant) the system becomes the EL equation:

$$\frac{4(n-1)}{n-2}\triangle_g u + h(g,\psi)u = f(\tau,\psi,V)u^{2^*-1} + \frac{\pi^2 + |\sigma|_g^2}{u^{2^*+1}}.$$

Here we assume that $(\pi,\sigma) \neq (0,0)$. The behavior of the set of solutions changes drastically according to the sign of f:

- When $f(\tau,\psi,V) \leq 0$ (defocusing), solutions are classified by the sub- and super-solution method (Isenberg '95) and are unique when they exist, except in one extreme case.
- When $f(\tau,\psi,V) > 0$ (focusing), the first existence result is in Hebey-Pacard-Pollack ('07). Solutions only exist for a limited set of coefficients, and there are generically two solutions (P., '14). Their bifurcation diagram can be complex, with multiple pitchfork bifurcations (Bizón-Pletka-Simon '15, Chruściel-Gicquaud '17)

An illustration: existence theory in the decoupled case

Under the assumption $\nabla \tau \equiv 0$ (the spacetime mean curvature is constant) the system becomes the EL equation:

$$\frac{4(n-1)}{n-2} \triangle_g u + h(g, \psi) u = f(\tau, \psi, V) u^{2^*-1} + \frac{\pi^2 + |\sigma|_g^2}{u^{2^*+1}}.$$

Here we assume that $(\pi, \sigma) \neq (0, 0)$. The behavior of the set of solutions changes drastically according to the sign of f:

- When $f(\tau, \psi, V) \leq 0$ (defocusing), solutions are classified by the sub- and super-solution method (Isenberg '95) and are unique when they exist, except in one extreme case.
- When $f(\tau, \psi, V) > 0$ (focusing), the first existence result is in Hebey-Pacard-Pollack ('07). Solutions only exist for a limited set of coefficients, and there are generically two solutions (P., '14). Their bifurcation diagram can be complex, with multiple pitchfork bifurcations (Bizón-Pletka-Simon '15, Chruściel-Gicquaud '17) and in pathological cases the equation can have an infinite number of (non-compact) solutions (P.-Wei, '15).

Existence theory in the decoupled case II

In the coupled case $\nabla\tau \not\equiv 0$, existence results are for instance:

- for the defocusing case $f \leq 0$: Maxwell ('04, '08), Holst-Nagy-Tsogtgerel ('08), Dahl-Gicquaud-Humbert ('13)

Existence theory in the decoupled case II

In the coupled case $\nabla \tau \not\equiv 0$, existence results are for instance:

- for the defocusing case $f \leq 0$: Maxwell ('04, '08), Holst-Nagy-Tsogtgerel ('08), Dahl-Gicquaud-Humbert ('13)
- for the focusing case $f > 0$: P. ('13), Gicquaud-Nguyen ('16), P. ('16)

Existence theory in the decoupled case II

In the coupled case $\nabla\tau \not\equiv 0$, existence results are for instance:

- for the defocusing case $f \le 0$: Maxwell ('04, '08), Holst-Nagy-Tsogtgerel ('08), Dahl-Gicquaud-Humbert ('13)
- for the focusing case $f > 0$: P. ('13), Gicquaud-Nguyen ('16), P. ('16)

The situation remains bad in general:

Existence theory in the decoupled case II

In the coupled case $\nabla\tau \not\equiv 0$, existence results are for instance:

- for the defocusing case $f \le 0$: Maxwell ('04, '08), Holst-Nagy-Tsogtgerel ('08), Dahl-Gicquaud-Humbert ('13)
- for the focusing case $f > 0$: P. ('13), Gicquaud-Nguyen ('16), P. ('16)

The situation remains bad in general: in the fully coupled case $\nabla\tau \not\equiv 0$ one loses uniqueness even in the defocusing case (Dilts-Holst-Kozareva-Maxwell, '18).

Existence theory in the decoupled case II

In the coupled case $\nabla\tau \not\equiv 0$, existence results are for instance:

- for the defocusing case $f \leq 0$: Maxwell ('04, '08), Holst-Nagy-Tsogtgerel ('08), Dahl-Gicquaud-Humbert ('13)
- for the focusing case $f > 0$: P. ('13), Gicquaud-Nguyen ('16), P. ('16)

The situation remains bad in general: in the fully coupled case $\nabla\tau \not\equiv 0$ one loses uniqueness even in the defocusing case (Dilts-Holst-Kozareva-Maxwell, '18).

When multiple solutions exist, it is often impossible to choose a preferred "physical solution".

Existence theory in the decoupled case II

In the coupled case $\nabla\tau \not\equiv 0$, existence results are for instance:

- for the defocusing case $f \leq 0$: Maxwell ('04, '08), Holst-Nagy-Tsogtgerel ('08), Dahl-Gicquaud-Humbert ('13)
- for the focusing case $f > 0$: P. ('13), Gicquaud-Nguyen ('16), P. ('16)

The situation remains bad in general: in the fully coupled case $\nabla\tau \not\equiv 0$ one loses uniqueness even in the defocusing case (Dilts-Holst-Kozareva-Maxwell, '18).

When multiple solutions exist, it is often impossible to choose a preferred "physical solution". In order to understand the solutions produced by the conformal method we therefore ask the following questions:

Existence theory in the decoupled case II

In the coupled case $\nabla\tau \not\equiv 0$, existence results are for instance:

- for the defocusing case $f \le 0$: Maxwell ('04, '08), Holst-Nagy-Tsogtgerel ('08), Dahl-Gicquaud-Humbert ('13)
- for the focusing case $f > 0$: P. ('13), Gicquaud-Nguyen ('16), P. ('16)

The situation remains bad in general: in the fully coupled case $\nabla\tau \not\equiv 0$ one loses uniqueness even in the defocusing case (Dilts-Holst-Kozareva-Maxwell, '18).

When multiple solutions exist, it is often impossible to choose a preferred "physical solution". In order to understand the solutions produced by the conformal method we therefore ask the following questions:

What can we say about the set of solutions of the Einstein-Lichnerowicz system?

Existence theory in the decoupled case II

In the coupled case $\nabla \tau \not\equiv 0$, existence results are for instance:

- for the defocusing case $f \le 0$: Maxwell ('04, '08), Holst-Nagy-Tsogtgerel ('08), Dahl-Gicquaud-Humbert ('13)
- for the focusing case $f > 0$: P. ('13), Gicquaud-Nguyen ('16), P. ('16)

The situation remains bad in general: in the fully coupled case $\nabla \tau \not\equiv 0$ one loses uniqueness even in the defocusing case (Dilts-Holst-Kozareva-Maxwell, '18).

When multiple solutions exist, it is often impossible to choose a preferred "physical solution". In order to understand the solutions produced by the conformal method we therefore ask the following questions:

What can we say about the set of solutions of the Einstein-Lichnerowicz system?

Is it compact in some strong topology, say $C^2(M)$?

Existence theory in the decoupled case II

In the coupled case $\nabla\tau \not\equiv 0$, existence results are for instance:

- for the defocusing case $f \leq 0$: Maxwell ('04, '08), Holst-Nagy-Tsogtgerel ('08), Dahl-Gicquaud-Humbert ('13)
- for the focusing case $f > 0$: P. ('13), Gicquaud-Nguyen ('16), P. ('16)

The situation remains bad in general: in the fully coupled case $\nabla\tau \not\equiv 0$ one loses uniqueness even in the defocusing case (Dilts-Holst-Kozareva-Maxwell, '18).

When multiple solutions exist, it is often impossible to choose a preferred "physical solution". In order to understand the solutions produced by the conformal method we therefore ask the following questions:

What can we say about the set of solutions of the Einstein-Lichnerowicz system?

Is it compact in some strong topology, say $C^2(M)$?

How does it behave with respect to perturbations of the physics data?

Existence theory in the decoupled case II

In the coupled case $\nabla \tau \not\equiv 0$, existence results are for instance:

- for the defocusing case $f \leq 0$: Maxwell ('04, '08), Holst-Nagy-Tsogtgerel ('08), Dahl-Gicquaud-Humbert ('13)
- for the focusing case $f > 0$: P. ('13), Gicquaud-Nguyen ('16), P. ('16)

The situation remains bad in general: in the fully coupled case $\nabla \tau \not\equiv 0$ one loses uniqueness even in the defocusing case (Dilts-Holst-Kozareva-Maxwell, '18).

When multiple solutions exist, it is often impossible to choose a preferred "physical solution". In order to understand the solutions produced by the conformal method we therefore ask the following questions:

What can we say about the set of solutions of the Einstein-Lichnerowicz system?

Is it compact in some strong topology, say $C^2(M)$?

How does it behave with respect to perturbations of the physics data?

We now introduce a notion of stability to answer these questions and investigate the "well-posedness" of the conformal method.

Elliptic stability: definition

Recall that if $\mathcal{D} = (V, \psi, \pi, \tau, \sigma)$ are physics data, the Einstein-Lichnerowicz system with physics data $\mathcal{D}$ is:

Elliptic stability: definition

Recall that if $\mathcal{D} = (V, \psi, \pi, \tau, \sigma)$ are physics data, the Einstein-Lichnerowicz system with physics data $\mathcal{D}$ is:

$$\begin{cases} \frac{4(n-1)}{n-2} \triangle_g u + h(g,\psi) u = f(\tau, \psi, V) u^{2^*-1} + \frac{\pi^2 + |\sigma + \mathcal{L}_g W|_g^2}{u^{2^*+1}} \ , \\ \overrightarrow{\triangle}_g W = -\frac{n-1}{n} u^{2^*} \nabla \tau - \pi \nabla \psi. \end{cases} \qquad (S_{\mathcal{D}})$$

Elliptic stability: definition

Recall that if $\mathcal{D} = (V, \psi, \pi, \tau, \sigma)$ are physics data, the Einstein-Lichnerowicz system with physics data $\mathcal{D}$ is:

$$\begin{cases} \frac{4(n-1)}{n-2} \triangle_g u + h(g,\psi) u = f(\tau, \psi, V) u^{2^*-1} + \frac{\pi^2 + |\sigma + \mathcal{L}_g W|_g^2}{u^{2^*+1}} , \\ \overrightarrow{\triangle}_g W = -\frac{n-1}{n} u^{2^*} \nabla \tau - \pi \nabla \psi. \end{cases} \qquad (S_{\mathcal{D}})$$

Definition ((Elliptic) Stability of system $S_{\mathcal{D}}$)

Elliptic stability: definition

Recall that if $\mathcal{D} = (V, \psi, \pi, \tau, \sigma)$ are physics data, the Einstein-Lichnerowicz system with physics data $\mathcal{D}$ is:

$$\begin{cases} \frac{4(n-1)}{n-2}\triangle_g u + h(g,\psi)u = f(\tau,\psi,V)u^{2^*-1} + \frac{\pi^2 + |\sigma + \mathcal{L}_g W|_g^2}{u^{2^*+1}}, \\ \overrightarrow{\triangle}_g W = -\frac{n-1}{n}u^{2^*}\nabla\tau - \pi\nabla\psi. \end{cases} \qquad (S_{\mathcal{D}})$$

Definition ((Elliptic) Stability of system $S_{\mathcal{D}}$)

Let $\mathcal{D} = (V, \psi, \pi, \tau, \sigma)$ be given physics data.

Elliptic stability: definition

Recall that if $\mathcal{D} = (V, \psi, \pi, \tau, \sigma)$ are physics data, the Einstein-Lichnerowicz system with physics data $\mathcal{D}$ is:

$$\begin{cases} \frac{4(n-1)}{n-2} \triangle_g u + h(g,\psi) u = f(\tau, \psi, V) u^{2^*-1} + \frac{\pi^2 + |\sigma + \mathcal{L}_g W|_g^2}{u^{2^*+1}} , \\ \overrightarrow{\triangle}_g W = -\frac{n-1}{n} u^{2^*} \nabla \tau - \pi \nabla \psi. \end{cases} \qquad (S_{\mathcal{D}})$$

Definition ((Elliptic) Stability of system $S_{\mathcal{D}}$)

Let $\mathcal{D} = (V, \psi, \pi, \tau, \sigma)$ be given physics data. Let $\mathcal{D}_\alpha = (V_\alpha, \psi_\alpha, \pi_\alpha, \tau_\alpha, \sigma_\alpha)_\alpha$ be a sequence of physics data converging to $\mathcal{D}$ in $C^2(M)$

Elliptic stability: definition

Recall that if $\mathcal{D} = (V, \psi, \pi, \tau, \sigma)$ are physics data, the Einstein-Lichnerowicz system with physics data $\mathcal{D}$ is:

$$\begin{cases} \dfrac{4(n-1)}{n-2} \triangle_g u + h(g,\psi) u = f(\tau, \psi, V) u^{2^*-1} + \dfrac{\pi^2 + |\sigma + \mathcal{L}_g W|_g^2}{u^{2^*+1}} , \\ \overrightarrow{\triangle}_g W = -\dfrac{n-1}{n} u^{2^*} \nabla \tau - \pi \nabla \psi. \end{cases} \qquad (S_{\mathcal{D}})$$

Definition ((Elliptic) Stability of system $S_{\mathcal{D}}$)

Let $\mathcal{D} = (V, \psi, \pi, \tau, \sigma)$ be given physics data. Let $\mathcal{D}_\alpha = (V_\alpha, \psi_\alpha, \pi_\alpha, \tau_\alpha, \sigma_\alpha)_\alpha$ be a sequence of physics data converging to $\mathcal{D}$ in $C^2(M)$ (the regularity can be weakened).

Elliptic stability: definition

Recall that if $\mathcal{D} = (V, \psi, \pi, \tau, \sigma)$ are physics data, the Einstein-Lichnerowicz system with physics data $\mathcal{D}$ is:

$$\begin{cases} \frac{4(n-1)}{n-2} \triangle_g u + h(g,\psi) u = f(\tau,\psi,V) u^{2^*-1} + \frac{\pi^2 + |\sigma + \mathcal{L}_g W|_g^2}{u^{2^*+1}} , \\ \overrightarrow{\triangle}_g W = -\frac{n-1}{n} u^{2^*} \nabla\tau - \pi \nabla \psi. \end{cases} \qquad (S_{\mathcal{D}})$$

Definition ((Elliptic) Stability of system $S_{\mathcal{D}}$)

Let $\mathcal{D} = (V, \psi, \pi, \tau, \sigma)$ be given physics data. Let $\mathcal{D}_\alpha = (V_\alpha, \psi_\alpha, \pi_\alpha, \tau_\alpha, \sigma_\alpha)_\alpha$ be a sequence of physics data converging to $\mathcal{D}$ in $C^2(M)$ (the regularity can be weakened).

We say that the system $S_{\mathcal{D}}$ is stable

Elliptic stability: definition

Recall that if $\mathcal{D} = (V, \psi, \pi, \tau, \sigma)$ are physics data, the Einstein-Lichnerowicz system with physics data $\mathcal{D}$ is:

$$\begin{cases} \frac{4(n-1)}{n-2} \triangle_g u + h(g, \psi) u = f(\tau, \psi, V) u^{2^*-1} + \frac{\pi^2 + |\sigma + \mathcal{L}_g W|_g^2}{u^{2^*+1}} , \\ \overrightarrow{\triangle}_g W = -\frac{n-1}{n} u^{2^*} \nabla \tau - \pi \nabla \psi. \end{cases} \qquad (S_{\mathcal{D}})$$

Definition ((Elliptic) Stability of system $S_{\mathcal{D}}$)

Let $\mathcal{D} = (V, \psi, \pi, \tau, \sigma)$ be given physics data. Let $\mathcal{D}_\alpha = (V_\alpha, \psi_\alpha, \pi_\alpha, \tau_\alpha, \sigma_\alpha)_\alpha$ be a sequence of physics data converging to $\mathcal{D}$ in $C^2(M)$ (the regularity can be weakened).

We say that the system $S_{\mathcal{D}}$ is stable if there exists $C_\theta > 0$ such that,

Elliptic stability: definition

Recall that if $\mathcal{D} = (V, \psi, \pi, \tau, \sigma)$ are physics data, the Einstein-Lichnerowicz system with physics data $\mathcal{D}$ is:

$$\begin{cases} \frac{4(n-1)}{n-2} \triangle_g u + h(g,\psi) u = f(\tau, \psi, V) u^{2^*-1} + \frac{\pi^2 + |\sigma + \mathcal{L}_g W|_g^2}{u^{2^*+1}} , \\ \overrightarrow{\triangle}_g W = -\frac{n-1}{n} u^{2^*} \nabla \tau - \pi \nabla \psi. \end{cases} \qquad (S_{\mathcal{D}})$$

Definition ((Elliptic) Stability of system $S_{\mathcal{D}}$)

Let $\mathcal{D} = (V, \psi, \pi, \tau, \sigma)$ be given physics data. Let $\mathcal{D}_\alpha = (V_\alpha, \psi_\alpha, \pi_\alpha, \tau_\alpha, \sigma_\alpha)_\alpha$ be a sequence of physics data converging to $\mathcal{D}$ in $C^2(M)$ (the regularity can be weakened).

We say that the system $S_{\mathcal{D}}$ is stable if there exists $C_\theta > 0$ such that, for any sequence $\mathcal{D}_\alpha$ converging towards $\mathcal{D}$,

Elliptic stability: definition

Recall that if $\mathcal{D} = (V, \psi, \pi, \tau, \sigma)$ are physics data, the Einstein-Lichnerowicz system with physics data $\mathcal{D}$ is:

$$\begin{cases} \frac{4(n-1)}{n-2} \triangle_g u + h(g,\psi) u = f(\tau, \psi, V) u^{2^*-1} + \frac{\pi^2 + |\sigma + \mathcal{L}_g W|_g^2}{u^{2^*+1}} , \\ \overrightarrow{\triangle}_g W = -\frac{n-1}{n} u^{2^*} \nabla \tau - \pi \nabla \psi. \end{cases} \qquad (S_{\mathcal{D}})$$

Definition ((Elliptic) Stability of system $S_{\mathcal{D}}$)

Let $\mathcal{D} = (V, \psi, \pi, \tau, \sigma)$ be given physics data. Let $\mathcal{D}_\alpha = (V_\alpha, \psi_\alpha, \pi_\alpha, \tau_\alpha, \sigma_\alpha)_\alpha$ be a sequence of physics data converging to $\mathcal{D}$ in $C^2(M)$ (the regularity can be weakened).

We say that the system $S_{\mathcal{D}}$ is stable if there exists $C_\theta > 0$ such that, for any sequence $\mathcal{D}_\alpha$ converging towards $\mathcal{D}$, and for any sequence $(u_\alpha, W_\alpha)_\alpha$ of solutions of $S_{\mathcal{D}_\alpha}$,

Elliptic stability: definition

Recall that if $\mathcal{D} = (V, \psi, \pi, \tau, \sigma)$ are physics data, the Einstein-Lichnerowicz system with physics data $\mathcal{D}$ is:

$$\begin{cases} \frac{4(n-1)}{n-2} \triangle_g u + h(g,\psi) u = f(\tau,\psi,V) u^{2^*-1} + \frac{\pi^2 + |\sigma + \mathcal{L}_g W|_g^2}{u^{2^*+1}}, \\ \overrightarrow{\triangle}_g W = -\frac{n-1}{n} u^{2^*} \nabla \tau - \pi \nabla \psi. \end{cases} \qquad (S_{\mathcal{D}})$$

Definition ((Elliptic) Stability of system $S_{\mathcal{D}}$)

Let $\mathcal{D} = (V, \psi, \pi, \tau, \sigma)$ be given physics data. Let $\mathcal{D}_\alpha = (V_\alpha, \psi_\alpha, \pi_\alpha, \tau_\alpha, \sigma_\alpha)_\alpha$ be a sequence of physics data converging to $\mathcal{D}$ in $C^2(M)$ (the regularity can be weakened).

We say that the system $S_{\mathcal{D}}$ is stable if there exists $C_\theta > 0$ such that, for any sequence $\mathcal{D}_\alpha$ converging towards $\mathcal{D}$, and for any sequence $(u_\alpha, W_\alpha)_\alpha$ of solutions of $S_{\mathcal{D}_\alpha}$,

$$\|u_\alpha\|_{C^{2,\theta}(M)} + \|\mathcal{L}_g W_\alpha\|_{C^{1,\theta}(M)} \le C \quad \text{for any } \alpha.$$

Equivalent formulations and remarks

- Saying that $S_{\mathcal{D}}$ is stable amounts to say that the set of solutions of $S_{\mathcal{D}}$ is *a priori* bounded in $C^{2,\theta}(M)$ for any $0 < \theta < 1$

Equivalent formulations and remarks

- Saying that $S_{\mathcal{D}}$ is stable amounts to say that the set of solutions of $S_{\mathcal{D}}$ is *a priori* bounded in $C^{2,\theta}(M)$ for any $0 < \theta < 1$ uniformly in the choice of its coefficients

Equivalent formulations and remarks

- Saying that $S_{\mathcal{D}}$ is stable amounts to say that the set of solutions of $S_{\mathcal{D}}$ is *a priori* bounded in $C^{2,\theta}(M)$ for any $0 < \theta < 1$ uniformly in the choice of its coefficients (that is, uniformly in the choice of $\mathcal{D}$).

Equivalent formulations and remarks

- Saying that $S_{\mathcal{D}}$ is stable amounts to say that the set of solutions of $S_{\mathcal{D}}$ is *a priori* bounded in $C^{2,\theta}(M)$ for any $0 < \theta < 1$ uniformly in the choice of its coefficients (that is, uniformly in the choice of $\mathcal{D}$).

  Stability implies compactness: if $S_{\mathcal{D}}$ is compact, its set of solutions is compact in the $C^2(M)$ topology.

Equivalent formulations and remarks

- Saying that $S_{\mathcal{D}}$ is stable amounts to say that the set of solutions of $S_{\mathcal{D}}$ is *a priori* bounded in $C^{2,\theta}(M)$ for any $0 < \theta < 1$ uniformly in the choice of its coefficients (that is, uniformly in the choice of $\mathcal{D}$).

  Stability implies compactness: if $S_{\mathcal{D}}$ is compact, its set of solutions is compact in the $C^2(M)$ topology.

- The stability of $S_{\mathcal{D}}$ implies in particular that any sequence $(u_\alpha, W_\alpha)_\alpha$ of solutions of Einstein-Lichnerowivz systems with perturbed physics data $\mathcal{D}_\alpha$

Equivalent formulations and remarks

- Saying that $S_{\mathcal{D}}$ is stable amounts to say that the set of solutions of $S_{\mathcal{D}}$ is *a priori* bounded in $C^{2,\theta}(M)$ for any $0 < \theta < 1$ uniformly in the choice of its coefficients (that is, uniformly in the choice of $\mathcal{D}$).

  Stability implies compactness: if $S_{\mathcal{D}}$ is compact, its set of solutions is compact in the $C^2(M)$ topology.

- The stability of $S_{\mathcal{D}}$ implies in particular that any sequence $(u_\alpha, W_\alpha)_\alpha$ of solutions of Einstein-Lichnerowivz systems with perturbed physics data $\mathcal{D}_\alpha$ converges in $C^2(M)$, up to a subsequence, to some solution of the limiting system $S_{\mathcal{D}}$.

Equivalent formulations and remarks

- Saying that $S_{\mathcal{D}}$ is stable amounts to say that the set of solutions of $S_{\mathcal{D}}$ is *a priori* bounded in $C^{2,\theta}(M)$ for any $0 < \theta < 1$ uniformly in the choice of its coefficients (that is, uniformly in the choice of $\mathcal{D}$).

  Stability implies compactness: if $S_{\mathcal{D}}$ is compact, its set of solutions is compact in the $C^2(M)$ topology.

- The stability of $S_{\mathcal{D}}$ implies in particular that any sequence $(u_\alpha, W_\alpha)_\alpha$ of solutions of Einstein-Lichnerowivz systems with perturbed physics data $\mathcal{D}_\alpha$ converges in $C^2(M)$, up to a subsequence, to some solution of the limiting system $S_{\mathcal{D}}$.

- In the definition we can only require *a priori* bounds on $\mathcal{L}_g W_\alpha$ (and not W_α)

Equivalent formulations and remarks

- Saying that $S_{\mathcal{D}}$ is stable amounts to say that the set of solutions of $S_{\mathcal{D}}$ is *a priori* bounded in $C^{2,\theta}(M)$ for any $0 < \theta < 1$ uniformly in the choice of its coefficients (that is, uniformly in the choice of $\mathcal{D}$).

  Stability implies compactness: if $S_{\mathcal{D}}$ is compact, its set of solutions is compact in the $C^2(M)$ topology.

- The stability of $S_{\mathcal{D}}$ implies in particular that any sequence $(u_\alpha, W_\alpha)_\alpha$ of solutions of Einstein-Lichnerowivz systems with perturbed physics data $\mathcal{D}_\alpha$ converges in $C^2(M)$, up to a subsequence, to some solution of the limiting system $S_{\mathcal{D}}$.

- In the definition we can only require *a priori* bounds on $\mathcal{L}_g W_\alpha$ (and not W_α) since the system is invariant up to adding to W_α a conformal Killing field (satisfying $\mathcal{L}_g Z = 0$).

Equivalent formulations and remarks

- Saying that $S_{\mathcal{D}}$ is stable amounts to say that the set of solutions of $S_{\mathcal{D}}$ is *a priori* bounded in $C^{2,\theta}(M)$ for any $0 < \theta < 1$ uniformly in the choice of its coefficients (that is, uniformly in the choice of $\mathcal{D}$).

  Stability implies compactness: if $S_{\mathcal{D}}$ is compact, its set of solutions is compact in the $C^2(M)$ topology.

- The stability of $S_{\mathcal{D}}$ implies in particular that any sequence $(u_\alpha, W_\alpha)_\alpha$ of solutions of Einstein-Lichnerowivz systems with perturbed physics data $\mathcal{D}_\alpha$ converges in $C^2(M)$, up to a subsequence, to some solution of the limiting system $S_{\mathcal{D}}$.

- In the definition we can only require *a priori* bounds on $\mathcal{L}_g W_\alpha$ (and not W_α) since the system is invariant up to adding to W_α a conformal Killing field (satisfying $\mathcal{L}_g Z = 0$).

In the second lecture we will state and prove some stability results.

Thank you for your attention!

(End of Lecture 1 and coffe break for everyone)

Contents

The definition again (a variant)

Definition ((Elliptic) Stability of system $S_{\mathcal{D}}$)

The definition again (a variant)

Definition ((Elliptic) Stability of system $S_{\mathcal{D}}$)

We say that the system $S_{\mathcal{D}}$ is stable

The definition again (a variant)

Definition ((Elliptic) Stability of system $S_{\mathcal{D}}$)

We say that the system $S_{\mathcal{D}}$ is stable if, for any $0 < \theta < 1$, there exists $C_\theta > 0$ and $\varepsilon_\theta > 0$

The definition again (a variant)

Definition ((Elliptic) Stability of system $S_{\mathcal{D}}$)

We say that the system $S_{\mathcal{D}}$ is stable if, for any $0 < \theta < 1$, there exists $C_\theta > 0$ and $\varepsilon_\theta > 0$ such that for any choice of physics data $\mathcal{D}'$ with

$$\|\mathcal{D} - \mathcal{D}'\|_{C^2(M)} < \varepsilon_\theta,$$

The definition again (a variant)

Definition ((Elliptic) Stability of system $S_{\mathcal{D}}$)

We say that the system $S_{\mathcal{D}}$ is stable if, for any $0 < \theta < 1$, there exists $C_\theta > 0$ and $\varepsilon_\theta > 0$ such that for any choice of physics data $\mathcal{D}'$ with

$$\|\mathcal{D} - \mathcal{D}'\|_{C^2(M)} < \varepsilon_\theta,$$

and for any solution (u, W) of the Einstein-Lichnerowicz $\mathcal{D}'$,

The definition again (a variant)

Definition ((Elliptic) Stability of system $S_{\mathcal{D}}$)

We say that the system $S_{\mathcal{D}}$ is stable if, for any $0 < \theta < 1$, there exists $C_\theta > 0$ and $\varepsilon_\theta > 0$ such that for any choice of physics data $\mathcal{D}'$ with

$$\|\mathcal{D} - \mathcal{D}'\|_{C^2(M)} < \varepsilon_\theta,$$

and for any solution (u, W) of the Einstein-Lichnerowicz $\mathcal{D}'$, we have

$$\|u\|_{C^{2,\theta}(M)} + \|\mathcal{L}_g W\|_{C^{1,\theta}(M)} \leq C_\theta.$$

The definition again (a variant)

Definition ((Elliptic) Stability of system $S_{\mathcal{D}}$)

We say that the system $S_{\mathcal{D}}$ is stable if, for any $0 < \theta < 1$, there exists $C_\theta > 0$ and $\varepsilon_\theta > 0$ such that for any choice of physics data $\mathcal{D}'$ with

$$\|\mathcal{D} - \mathcal{D}'\|_{C^2(M)} < \varepsilon_\theta,$$

and for any solution (u, W) of the Einstein-Lichnerowicz $\mathcal{D}'$, we have

$$\|u\|_{C^{2,\theta}(M)} + \|\mathcal{L}_g W\|_{C^{1,\theta}(M)} \le C_\theta.$$

The statement of the stability results (and their proofs) again heavily depend on the focusing or defocusing setting.

Compactness in the defocusing case

In the defocusing vacuum case, a compactness result is available:

Compactness in the defocusing case

In the defocusing vacuum case, a compactness result is available:

Theorem (Dahl-Gicquaud-Humbert '13)

Assume that $\tau \neq 0$ and (M, g) has no conformal Killing fields.

Compactness in the defocusing case

In the defocusing vacuum case, a compactness result is available:

Theorem (Dahl-Gicquaud-Humbert '13)

Assume that $\tau \neq 0$ *and* (M, g) *has no conformal Killing fields. Assume that for any* $\varepsilon \in (0, 1]$ *the following equation in* M^n *admits no non-trivial solutions* $W \in \bigwedge^1(T^*M)$*:*

Compactness in the defocusing case

In the defocusing vacuum case, a compactness result is available:

Theorem (Dahl-Gicquaud-Humbert '13)

Assume that $\tau \neq 0$ and (M, g) has no conformal Killing fields. Assume that for any $\varepsilon \in (0, 1]$ the following equation in M^n admits no non-trivial solutions $W \in \bigwedge^1(T^*M)$:

$$\overrightarrow{\triangle}_g W = \varepsilon \sqrt{\frac{n-1}{n}} |\mathcal{L}_g W| \frac{d\tau}{\tau}.$$

Compactness in the defocusing case

In the defocusing vacuum case, a compactness result is available:

Theorem (Dahl-Gicquaud-Humbert '13)

Assume that $\tau \neq 0$ and (M, g) has no conformal Killing fields. Assume that for any $\varepsilon \in (0, 1]$ the following equation in M^n admits no non-trivial solutions $W \in \bigwedge^1(T^*M)$:

$$\overrightarrow{\triangle}_g W = \varepsilon \sqrt{\frac{n-1}{n}} |\mathcal{L}_g W| \frac{d\tau}{\tau}.$$

Then the set of solutions of the vacuum Einstein-Lichnerowicz system is non-empty and compact in $C^2(M)$.

Sketch of the proof in the defocusing case

Let (u_i, W_i) be a sequence of solutions of the vacuum constraint system:

Sketch of the proof in the defocusing case

Let (u_i, W_i) be a sequence of solutions of the vacuum constraint system:

$$\begin{cases} \frac{4(n-1)}{n-2}\triangle_g u_i + R(g)u_i = -\frac{n-1}{n}\tau^2 u_i^{2^*-1} + \frac{|\sigma + \mathcal{L}_g W_i|_g^2}{u_i^{2^*+1}} , \\ \overrightarrow{\triangle}_g W_i = -\frac{n-1}{n} u_i^{2^*-\delta_i}\nabla\tau \end{cases}$$

Sketch of the proof in the defocusing case

Let (u_i, W_i) be a sequence of solutions of the vacuum constraint system:

$$\begin{cases} \frac{4(n-1)}{n-2}\triangle_g u_i + R(g)u_i = -\frac{n-1}{n}\tau^2 u_i^{2^*-1} + \frac{|\sigma + \mathcal{L}_g W_i|_g^2}{u_i^{2^*+1}} \ , \\ \overrightarrow{\triangle}_g W_i = -\frac{n-1}{n} u_i^{2^*-\delta_i}\nabla\tau \end{cases}$$

where $\delta_i \to 0$, $\delta_i > 0$. (Remember: $2^* = \frac{2n}{n-2}$.)

Sketch of the proof in the defocusing case

Let (u_i, W_i) be a sequence of solutions of the vacuum constraint system:

$$\begin{cases} \frac{4(n-1)}{n-2}\triangle_g u_i + R(g)u_i = -\frac{n-1}{n}\tau^2 u_i^{2^*-1} + \frac{|\sigma + \mathcal{L}_g W_i|_g^2}{u_i^{2^*+1}} \;, \\ \overrightarrow{\triangle}_g W_i = -\frac{n-1}{n} u_i^{2^*-\delta_i} \nabla\tau \end{cases}$$

where $\delta_i \to 0$, $\delta_i > 0$. (Remember: $2^* = \frac{2n}{n-2}$.)

Assume that u_i and W_i blow-up,

Sketch of the proof in the defocusing case

Let (u_i, W_i) be a sequence of solutions of the vacuum constraint system:

$$\begin{cases} \dfrac{4(n-1)}{n-2}\triangle_g u_i + R(g)u_i = -\dfrac{n-1}{n}\tau^2 u_i^{2^*-1} + \dfrac{|\sigma + \mathcal{L}_g W_i|_g^2}{u_i^{2^*+1}} \ , \\ \overrightarrow{\triangle}_g W_i = -\dfrac{n-1}{n} u_i^{2^*-\delta_i} \nabla\tau \end{cases}$$

where $\delta_i \to 0$, $\delta_i > 0$. (Remember: $2^* = \frac{2n}{n-2}$.)

Assume that u_i and W_i blow-up, let $\gamma_i = \|\mathcal{L}_g W_i\|_{L^2(M)} \to +\infty$.

Sketch of the proof in the defocusing case

Let (u_i, W_i) be a sequence of solutions of the vacuum constraint system:

$$\begin{cases} \dfrac{4(n-1)}{n-2}\triangle_g u_i + R(g)u_i = -\dfrac{n-1}{n}\tau^2 u_i^{2^*-1} + \dfrac{|\sigma + \mathcal{L}_g W_i|_g^2}{u_i^{2^*+1}} \ , \\ \overrightarrow{\triangle}_g W_i = -\dfrac{n-1}{n} u_i^{2^*-\delta_i}\nabla\tau \end{cases}$$

where $\delta_i \to 0$, $\delta_i > 0$. (Remember: $2^* = \frac{2n}{n-2}$.)

Assume that u_i and W_i blow-up, let $\gamma_i = \|\mathcal{L}_g W_i\|_{L^2(M)} \to +\infty$. Comparison and bootstrap arguments show that $\gamma_i^{-1} W_i$ and $\gamma_i^{-\frac{n-2}{4n}} u_i$ converge,

Sketch of the proof in the defocusing case

Let (u_i, W_i) be a sequence of solutions of the vacuum constraint system:

$$\begin{cases} \frac{4(n-1)}{n-2} \triangle_g u_i + R(g) u_i = -\frac{n-1}{n} \tau^2 u_i^{2^*-1} + \frac{|\sigma + \mathcal{L}_g W_i|_g^2}{u_i^{2^*+1}} , \\ \overrightarrow{\triangle}_g W_i = -\frac{n-1}{n} u_i^{2^*-\delta_i} \nabla \tau \end{cases}$$

where $\delta_i \to 0$, $\delta_i > 0$. (Remember: $2^* = \frac{2n}{n-2}$.)

Assume that u_i and W_i blow-up, let $\gamma_i = \|\mathcal{L}_g W_i\|_{L^2(M)} \to +\infty$. Comparison and bootstrap arguments show that $\gamma_i^{-1} W_i$ and $\gamma_i^{-\frac{n-2}{4n}} u_i$ converge, respectively in in $C^1(M)$ and $L^\infty(M)$,

Sketch of the proof in the defocusing case

Let (u_i, W_i) be a sequence of solutions of the vacuum constraint system:

$$\begin{cases} \frac{4(n-1)}{n-2}\triangle_g u_i + R(g)u_i = -\frac{n-1}{n}\tau^2 u_i^{2^*-1} + \frac{|\sigma + \mathcal{L}_g W_i|_g^2}{u_i^{2^*+1}} , \\ \overrightarrow{\triangle}_g W_i = -\frac{n-1}{n} u_i^{2^*-\delta_i} \nabla\tau \end{cases}$$

where $\delta_i \to 0$, $\delta_i > 0$. (Remember: $2^* = \frac{2n}{n-2}$.)

Assume that u_i and W_i blow-up, let $\gamma_i = \|\mathcal{L}_g W_i\|_{L^2(M)} \to +\infty$. Comparison and bootstrap arguments show that $\gamma_i^{-1} W_i$ and $\gamma_i^{-\frac{n-2}{4n}} u_i$ converge, respectively in in $C^1(M)$ and $L^\infty(M)$, towards W_∞ and u_∞.

Sketch of the proof in the defocusing case

Let (u_i, W_i) be a sequence of solutions of the vacuum constraint system:

$$\begin{cases} \dfrac{4(n-1)}{n-2}\triangle_g u_i + R(g)u_i = -\dfrac{n-1}{n}\tau^2 u_i^{2^*-1} + \dfrac{|\sigma + \mathcal{L}_g W_i|_g^2}{u_i^{2^*+1}} , \\ \overrightarrow{\triangle}_g W_i = -\dfrac{n-1}{n} u_i^{2^*-\delta_i}\nabla\tau \end{cases}$$

where $\delta_i \to 0$, $\delta_i > 0$. (Remember: $2^* = \frac{2n}{n-2}$.)

Assume that u_i and W_i blow-up, let $\gamma_i = \|\mathcal{L}_g W_i\|_{L^2(M)} \to +\infty$. Comparison and bootstrap arguments show that $\gamma_i^{-1} W_i$ and $\gamma_i^{-\frac{n-2}{4n}} u_i$ converge, respectively in in $C^1(M)$ and $L^\infty(M)$, towards W_∞ and u_∞. And that $W_\infty \not\equiv 0$ satisfies:

$$\overrightarrow{\triangle}_g W_\infty = \varepsilon\sqrt{\frac{n-1}{n}}|\mathcal{L}_g W_\infty|\frac{d\tau}{\tau}$$

Sketch of the proof in the defocusing case

Let (u_i, W_i) be a sequence of solutions of the vacuum constraint system:

$$\begin{cases} \frac{4(n-1)}{n-2} \triangle_g u_i + R(g) u_i = -\frac{n-1}{n} \tau^2 u_i^{2^*-1} + \frac{|\sigma + \mathcal{L}_g W_i|_g^2}{u_i^{2^*+1}} , \\ \overrightarrow{\triangle}_g W_i = -\frac{n-1}{n} u_i^{2^* - \delta_i} \nabla \tau \end{cases}$$

where $\delta_i \to 0$, $\delta_i > 0$. (Remember: $2^* = \frac{2n}{n-2}$.)

Assume that u_i and W_i blow-up, let $\gamma_i = \|\mathcal{L}_g W_i\|_{L^2(M)} \to +\infty$. Comparison and bootstrap arguments show that $\gamma_i^{-1} W_i$ and $\gamma_i^{-\frac{n-2}{4n}} u_i$ converge, respectively in in $C^1(M)$ and $L^\infty(M)$, towards W_∞ and u_∞. And that $W_\infty \not\equiv 0$ satisfies:

$$\overrightarrow{\triangle}_g W_\infty = \varepsilon \sqrt{\frac{n-1}{n}} |\mathcal{L}_g W_\infty| \frac{d\tau}{\tau}$$

where $\varepsilon = \lim \gamma_i^{-c(n)\varepsilon_i}$ (up to a constant).

Sketch of the proof in the defocusing case

Let (u_i, W_i) be a sequence of solutions of the vacuum constraint system:

$$\begin{cases} \frac{4(n-1)}{n-2}\triangle_g u_i + R(g)u_i = -\frac{n-1}{n}\tau^2 u_i^{2^*-1} + \frac{|\sigma + \mathcal{L}_g W_i|_g^2}{u_i^{2^*+1}} \,, \\ \overrightarrow{\triangle}_g W_i = -\frac{n-1}{n} u_i^{2^*-\delta_i} \nabla\tau \end{cases}$$

where $\delta_i \to 0$, $\delta_i > 0$. (Remember: $2^* = \frac{2n}{n-2}$.)

Assume that u_i and W_i blow-up, let $\gamma_i = \|\mathcal{L}_g W_i\|_{L^2(M)} \to +\infty$. Comparison and bootstrap arguments show that $\gamma_i^{-1} W_i$ and $\gamma_i^{-\frac{n-2}{4n}} u_i$ converge, respectively in in $C^1(M)$ and $L^\infty(M)$, towards W_∞ and u_∞. And that $W_\infty \not\equiv 0$ satisfies:

$$\overrightarrow{\triangle}_g W_\infty = \varepsilon\sqrt{\frac{n-1}{n}}|\mathcal{L}_g W_\infty|\frac{d\tau}{\tau}$$

where $\varepsilon = \lim \gamma_i^{-c(n)\varepsilon_i}$ (up to a constant). Blow-up is global here:

$$W_i \sim \gamma_i(W_\infty + o(1)), \quad u_i \sim \gamma_i^{\frac{n-2}{4n}}(u_\infty + o(1)) \quad \text{in } M$$

Sketch of the proof in the defocusing case

Let (u_i, W_i) be a sequence of solutions of the vacuum constraint system:

$$\begin{cases} \frac{4(n-1)}{n-2}\triangle_g u_i + R(g)u_i = -\frac{n-1}{n}\tau^2 u_i^{2^*-1} + \frac{|\sigma + \mathcal{L}_g W_i|_g^2}{u_i^{2^*+1}}, \\ \overrightarrow{\triangle}_g W_i = -\frac{n-1}{n} u_i^{2^*-\delta_i}\nabla\tau \end{cases}$$

where $\delta_i \to 0$, $\delta_i > 0$. (Remember: $2^* = \frac{2n}{n-2}$.)

Assume that u_i and W_i blow-up, let $\gamma_i = \|\mathcal{L}_g W_i\|_{L^2(M)} \to +\infty$. Comparison and bootstrap arguments show that $\gamma_i^{-1} W_i$ and $\gamma_i^{-\frac{n-2}{4n}} u_i$ converge, respectively in in $C^1(M)$ and $L^\infty(M)$, towards W_∞ and u_∞. And that $W_\infty \not\equiv 0$ satisfies:

$$\overrightarrow{\triangle}_g W_\infty = \varepsilon\sqrt{\frac{n-1}{n}}|\mathcal{L}_g W_\infty|\frac{d\tau}{\tau}$$

where $\varepsilon = \lim \gamma_i^{-c(n)\varepsilon_i}$ (up to a constant). Blow-up is global here:

$$W_i \sim \gamma_i(W_\infty + o(1)), \quad u_i \sim \gamma_i^{\frac{n-2}{4n}}(u_\infty + o(1)) \quad \text{in } M$$

and is governed by a limiting equation in M.

Elliptic Stabiliy: the results in the focusing case

The situation is different in the focusing case:

Elliptic Stabiliy: the results in the focusing case

The situation is different in the focusing case:

Theorem (Druet-P., '14 – P., '15)

Elliptic Stabiliy: the results in the focusing case

The situation is different in the focusing case:

Theorem (Druet-P., '14 – P., '15)

Let (M^n, g) be a closed locally conformally flat manifold of dimension $n \geq 3$. Let $\mathcal{D} = (V, \psi, \pi, \tau, \sigma)$ be some focusing physics data, i.e. with $f(\tau, \psi, V) > 0$.

Elliptic Stabiliy: the results in the focusing case

The situation is different in the focusing case:

Theorem (Druet-P., '14 – P., '15)

Let (M^n, g) be a closed locally conformally flat manifold of dimension $n \geq 3$. Let $\mathcal{D} = (V, \psi, \pi, \tau, \sigma)$ be some focusing physics data, i.e. with $f(\tau, \psi, V) > 0$. The Einstein-Lichnerowicz system $S_{\mathcal{D}}$ is stable under the following assumptions:

Elliptic Stabiliy: the results in the focusing case

The situation is different in the focusing case:

Theorem (Druet-P., '14 – P., '15)

Let (M^n, g) be a closed locally conformally flat manifold of dimension $n \geq 3$. Let $\mathcal{D} = (V, \psi, \pi, \tau, \sigma)$ be some focusing physics data, i.e. with $f(\tau, \psi, V) > 0$. The Einstein-Lichnerowicz system $S_{\mathcal{D}}$ is stable under the following assumptions:

- $\pi \not\equiv 0$ *if* $3 \leq n \leq 5$,

Elliptic Stabiliy: the results in the focusing case

The situation is different in the focusing case:

Theorem (Druet-P., '14 – P., '15)

Let (M^n, g) be a closed locally conformally flat manifold of dimension $n \geq 3$. Let $\mathcal{D} = (V, \psi, \pi, \tau, \sigma)$ be some focusing physics data, i.e. with $f(\tau, \psi, V) > 0$. The Einstein-Lichnerowicz system $S_{\mathcal{D}}$ is stable under the following assumptions:

- $\pi \not\equiv 0$ *if* $3 \leq n \leq 5$,
- $\pi \not\equiv 0$

Elliptic Stabiliy: the results in the focusing case

The situation is different in the focusing case:

Theorem (Druet-P., '14 – P., '15)

Let (M^n, g) be a closed locally conformally flat manifold of dimension $n \geq 3$. Let $\mathcal{D} = (V, \psi, \pi, \tau, \sigma)$ be some focusing physics data, i.e. with $f(\tau, \psi, V) > 0$. The Einstein-Lichnerowicz system $S_{\mathcal{D}}$ is stable under the following assumptions:

- $\pi \not\equiv 0$ *if* $3 \leq n \leq 5$,
- $\pi \not\equiv 0$ *and* ψ *and* τ *have no common critical points if* $n \geq 6$.

.

Elliptic Stabiliy: the results in the focusing case

The situation is different in the focusing case:

Theorem (Druet-P., '14 – P., '15)

Let (M^n, g) be a closed locally conformally flat manifold of dimension $n \geq 3$. Let $\mathcal{D} = (V, \psi, \pi, \tau, \sigma)$ be some focusing physics data, i.e. with $f(\tau, \psi, V) > 0$. The Einstein-Lichnerowicz system $S_{\mathcal{D}}$ is stable under the following assumptions:

- $\pi \not\equiv 0$ *if* $3 \leq n \leq 5$,
- $\pi \not\equiv 0$ *and* ψ *and* τ *have no common critical points if* $n \geq 6$.

.

Generalizes previous results for the decoupled case by Druet-Hebey ('08) and Hebey-Veronelli ('12).

Elliptic Stabiliy: the results in the focusing case

The situation is different in the focusing case:

Theorem (Druet-P., '14 – P., '15)

Let (M^n, g) be a closed locally conformally flat manifold of dimension $n \geq 3$. Let $\mathcal{D} = (V, \psi, \pi, \tau, \sigma)$ be some focusing physics data, i.e. with $f(\tau, \psi, V) > 0$. The Einstein-Lichnerowicz system $S_{\mathcal{D}}$ is stable under the following assumptions:

- $\pi \not\equiv 0$ *if* $3 \leq n \leq 5$,
- $\pi \not\equiv 0$ *and* ψ *and* τ *have no common critical points if* $n \geq 6$.

.

Generalizes previous results for the decoupled case by Druet-Hebey ('08) and Hebey-Veronelli ('12).

The Theorem covers the setting of a positive cosmological constant $V \equiv \Lambda > 0$ (e.g. in the maximal case $\tau \equiv 0$).

Elliptic Stabiliy: the results in the focusing case

The situation is different in the focusing case:

Theorem (Druet-P., '14 – P., '15)

Let (M^n, g) be a closed locally conformally flat manifold of dimension $n \geq 3$. Let $\mathcal{D} = (V, \psi, \pi, \tau, \sigma)$ be some focusing physics data, i.e. with $f(\tau, \psi, V) > 0$. The Einstein-Lichnerowicz system $S_{\mathcal{D}}$ is stable under the following assumptions:

- $\pi \not\equiv 0$ *if* $3 \leq n \leq 5$,
- $\pi \not\equiv 0$ *and* ψ *and* τ *have no common critical points if* $n \geq 6$.

.

Generalizes previous results for the decoupled case by Druet-Hebey ('08) and Hebey-Veronelli ('12).

The Theorem covers the setting of a positive cosmological constant $V \equiv \Lambda > 0$ (e.g. in the maximal case $\tau \equiv 0$). The vacuum case $f(\tau, \psi, V) \leq 0$

Elliptic Stabiliy: the results in the focusing case

The situation is different in the focusing case:

Theorem (Druet-P., '14 – P., '15)

Let (M^n, g) be a closed locally conformally flat manifold of dimension $n \geq 3$. Let $\mathcal{D} = (V, \psi, \pi, \tau, \sigma)$ be some focusing physics data, i.e. with $f(\tau, \psi, V) > 0$. The Einstein-Lichnerowicz system $S_{\mathcal{D}}$ is stable under the following assumptions:

- $\pi \not\equiv 0$ *if* $3 \leq n \leq 5$,
- $\pi \not\equiv 0$ *and* ψ *and* τ *have no common critical points if* $n \geq 6$.

.

Generalizes previous results for the decoupled case by Druet-Hebey ('08) and Hebey-Veronelli ('12).

The Theorem covers the setting of a positive cosmological constant $V \equiv \Lambda > 0$ (e.g. in the maximal case $\tau \equiv 0$). The vacuum case $f(\tau, \psi, V) \leq 0$ is not covered here. Matter creates stability for the Einstein-Lichnerowicz system.

A remark on the assumptions

The assumptions of the Theorem are generic in the choice of $\mathcal{D}$.

A remark on the assumptions

The assumptions of the Theorem are generic in the choice of $\mathcal{D}$. As soon as they are not met, pathological phenomena (= non compact sequences of solutions) occur (P.-Wei '15, P. '16).

A remark on the assumptions

The assumptions of the Theorem are generic in the choice of $\mathcal{D}$. As soon as they are not met, pathological phenomena (= non compact sequences of solutions) occur (P.-Wei '15, P. '16).

Conditions ensuring the stability of the system are local here.

A remark on the assumptions

The assumptions of the Theorem are generic in the choice of $\mathcal{D}$. As soon as they are not met, pathological phenomena (= non compact sequences of solutions) occur (P.-Wei '15, P. '16).

Conditions ensuring the stability of the system are local here. As we shall see in the proof, they occur in order to rule out concentration phenomena.

A remark on the assumptions

The assumptions of the Theorem are generic in the choice of $\mathcal{D}$. As soon as they are not met, pathological phenomena (= non compact sequences of solutions) occur (P.-Wei '15, P. '16).

Conditions ensuring the stability of the system are local here. As we shall see in the proof, they occur in order to rule out concentration phenomena. For this same reason we also need convergence of $\mathcal{D}$ in $C^2(M)$

A remark on the assumptions

The assumptions of the Theorem are generic in the choice of $\mathcal{D}$. As soon as they are not met, pathological phenomena (= non compact sequences of solutions) occur (P.-Wei '15, P. '16).

Conditions ensuring the stability of the system are local here. As we shall see in the proof, they occur in order to rule out concentration phenomena. For this same reason we also need convergence of $\mathcal{D}$ in $C^2(M)$ (the convergence can be lowered according to the dimension).

The proof: understanding concentration phenomena

The proof proceeds by contradiction. Let $(u_\alpha, W_\alpha)_\alpha$ be a sequence of solutions of the system and assume that it blows-up:

The proof: understanding concentration phenomena

The proof proceeds by contradiction. Let $(u_\alpha, W_\alpha)_\alpha$ be a sequence of solutions of the system and assume that it blows-up:

$$\|u\|_{C^0(M)} + \|\mathcal{L}_g W\|_{C^0(M)} \to +\infty$$

as $\alpha \to +\infty$.

The proof: understanding concentration phenomena

The proof proceeds by contradiction. Let $(u_\alpha, W_\alpha)_\alpha$ be a sequence of solutions of the system and assume that it blows-up:

$$\|u\|_{C^0(M)} + \|\mathcal{L}_g W\|_{C^0(M)} \to +\infty$$

as $\alpha \to +\infty$. These norms are weaker than the ones in the stability definition, but it's enough to consider them by elliptic theory.

The proof: understanding concentration phenomena

The proof proceeds by contradiction. Let $(u_\alpha, W_\alpha)_\alpha$ be a sequence of solutions of the system and assume that it blows-up:

$$\|u\|_{C^0(M)} + \|\mathcal{L}_g W\|_{C^0(M)} \to +\infty$$

as $\alpha \to +\infty$. These norms are weaker than the ones in the stability definition, but it's enough to consider them by elliptic theory.

Let $\mu_\alpha > 0$, $\mu_\alpha \to 0$ be defined by:

$$\mu_\alpha^{-n} := \max_M \left(u_\alpha^{\frac{2n}{n-2}} + |\mathcal{L}_g W_\alpha|_g \right).$$

The proof: understanding concentration phenomena

The proof proceeds by contradiction. Let $(u_\alpha, W_\alpha)_\alpha$ be a sequence of solutions of the system and assume that it blows-up:

$$\|u\|_{C^0(M)} + \|\mathcal{L}_g W\|_{C^0(M)} \to +\infty$$

as $\alpha \to +\infty$. These norms are weaker than the ones in the stability definition, but it's enough to consider them by elliptic theory.

Let $\mu_\alpha > 0$, $\mu_\alpha \to 0$ be defined by:

$$\mu_\alpha^{-n} := \max_M \left(u_\alpha^{\frac{2n}{n-2}} + |\mathcal{L}_g W_\alpha|_g \right).$$

If x_α is the point where this maximum is attained, we let, for $x \in \mathbb{R}^n$:

$$v_\alpha(x) = \mu_\alpha^{\frac{n-2}{2}} \exp_{x_\alpha}^* u_\alpha(\mu_\alpha x) \quad \text{and} \quad X_\alpha(x) = \mu_\alpha^{n-1} \exp_{x_\alpha}^* W_\alpha(\mu_\alpha x)$$

be the rescalings of u_α and W_α at distances comparable to μ_α to x_α.

We can then prove that v_α and X_α converge in $C^{1,\eta}_{loc}(\mathbb{R}^n)$, towards non-zero solutions (U, X) of

$$\begin{cases} \triangle_\xi U = f(x_0) U^{2^*-1}, \\ \overrightarrow{\triangle}_g X = 0. \end{cases}$$

We can then prove that v_α and X_α converge in $C^{1,\eta}_{loc}(\mathbb{R}^n)$, towards non-zero solutions (U, X) of

$$\begin{cases} \triangle_\xi U = f(x_0) U^{2^*-1}, \\ \overrightarrow{\triangle_g} X = 0. \end{cases}$$

where $x_0 = \lim x_\alpha$.

We can then prove that v_α and X_α converge in $C^{1,\eta}_{loc}(\mathbb{R}^n)$, towards non-zero solutions (U, X) of

$$\begin{cases} \triangle_\xi U = f(x_0) U^{2^*-1}, \\ \overrightarrow{\triangle_g} X = 0. \end{cases}$$

where $x_0 = \lim x_\alpha$. (The convergence of X_α is highly non-trivial and follows from the structure of the system).

We can then prove that v_α and X_α converge in $C^{1,\eta}_{loc}(\mathbb{R}^n)$, towards non-zero solutions (U, X) of

$$\begin{cases} \triangle_\xi U = f(x_0) U^{2^*-1}, \\ \overrightarrow{\triangle_g} X = 0. \end{cases}$$

where $x_0 = \lim x_\alpha$. (The convergence of X_α is highly non-trivial and follows from the structure of the system).

Solutions of the first equation are classified, and this describes the local behavior around some (actually, any) concentration point.

We can then prove that v_α and X_α converge in $C^{1,\eta}_{loc}(\mathbb{R}^n)$, towards non-zero solutions (U, X) of

$$\begin{cases} \triangle_\xi U = f(x_0) U^{2^*-1}, \\ \overrightarrow{\triangle_g} X = 0. \end{cases}$$

where $x_0 = \lim x_\alpha$. (The convergence of X_α is highly non-trivial and follows from the structure of the system).

Solutions of the first equation are classified, and this describes the local behavior around some (actually, any) concentration point.

The proof then consists in understanding the mutual interactions of the different concentration points.

Step 1: Detection of blow-up points

Proposition

There exists a sequence $(N_\alpha)_\alpha$, $N_\alpha \geq 2$ *and sequences of points* $(x_{1,\alpha}, \ldots, x_{N_\alpha,\alpha})_\alpha$ *in* M *such that:*

Step 1: Detection of blow-up points

Proposition

There exists a sequence $(N_\alpha)_\alpha$, $N_\alpha \geq 2$ *and sequences of points* $(x_{1,\alpha}, \ldots, x_{N_\alpha,\alpha})_\alpha$ *in* M *such that:*

- $\nabla u_\alpha(x_{i,\alpha}) = 0$ *for* $1 \leq i \leq N_\alpha$,

Step 1: Detection of blow-up points

Proposition

There exists a sequence $(N_\alpha)_\alpha$, $N_\alpha \geq 2$ *and sequences of points* $(x_{1,\alpha}, \dots, x_{N_\alpha,\alpha})_\alpha$ *in* M *such that:*

- $\nabla u_\alpha(x_{i,\alpha}) = 0$ *for* $1 \leq i \leq N_\alpha$,
- $d_g(x_{i,\alpha}, x_{j,\alpha})^{\frac{1}{2}} u_\alpha(x_{i,\alpha}) \geq 1$ *for* $i, j \in \{1, \dots, N_\alpha\}, i \neq j$ *and*

Step 1: Detection of blow-up points

Proposition

There exists a sequence $(N_\alpha)_\alpha$, $N_\alpha \geq 2$ *and sequences of points* $(x_{1,\alpha}, \dots, x_{N_\alpha,\alpha})_\alpha$ *in* M *such that:*

- $\nabla u_\alpha(x_{i,\alpha}) = 0$ *for* $1 \leq i \leq N_\alpha$,
- $d_g(x_{i,\alpha}, x_{j,\alpha})^{\frac{1}{2}} u_\alpha(x_{i,\alpha}) \geq 1$ *for* $i, j \in \{1, \dots, N_\alpha\}, i \neq j$ *and*
- *for any* $x \in M$:

$$\left(\min_{i=1,\dots,N_\alpha} d_g(x_{i,\alpha}, x) \right)^n \left(u_\alpha(x)^{2^*} + |\mathcal{L}_g W_\alpha|_g(x) \right) \leq C.$$

Step 1: Detection of blow-up points

Proposition

There exists a sequence $(N_\alpha)_\alpha$, $N_\alpha \geq 2$ *and sequences of points* $(x_{1,\alpha}, \dots, x_{N_\alpha,\alpha})_\alpha$ *in* M *such that:*

- $\nabla u_\alpha(x_{i,\alpha}) = 0$ *for* $1 \leq i \leq N_\alpha$,
- $d_g(x_{i,\alpha}, x_{j,\alpha})^{\frac{1}{2}} u_\alpha(x_{i,\alpha}) \geq 1$ *for* $i, j \in \{1, \dots, N_\alpha\}, i \neq j$ *and*
- *for any* $x \in M$*:*

$$\left(\min_{i=1,\dots,N_\alpha} d_g\left(x_{i,\alpha}, x\right) \right)^n \left(u_\alpha(x)^{2^*} + \left|\mathcal{L}_g W_\alpha\right|_g(x) \right) \leq C.$$

The number N_α can *a priori* go to $+\infty$ since we did not assume anything on the energy of (u_α, W_α).

Step 1: Detection of blow-up points

Proposition

There exists a sequence $(N_\alpha)_\alpha$, $N_\alpha \geq 2$ *and sequences of points* $(x_{1,\alpha}, \dots, x_{N_\alpha,\alpha})_\alpha$ *in* M *such that:*

- $\nabla u_\alpha(x_{i,\alpha}) = 0$ *for* $1 \leq i \leq N_\alpha$,
- $d_g(x_{i,\alpha}, x_{j,\alpha})^{\frac{1}{2}} u_\alpha(x_{i,\alpha}) \geq 1$ *for* $i, j \in \{1, \dots, N_\alpha\}, i \neq j$ *and*
- *for any* $x \in M$*:*

$$\left(\min_{i=1,\dots,N_\alpha} d_g\left(x_{i,\alpha}, x\right)\right)^n \left(u_\alpha(x)^{2^*} + \left|\mathcal{L}_g W_\alpha\right|_g (x)\right) \leq C.$$

The number N_α can *a priori* go to $+\infty$ since we did not assume anything on the energy of (u_α, W_α). We expect $(x_{1,\alpha}, \dots, x_{N_\alpha,\alpha})$ to exhaust the set of concentration points.

Step 2: Sharp asymptotics around a concentration point

Denote by $\rho_{i,\alpha}$ the maximum radius around which the concentration point $x_{i,\alpha}$ is dominant.

Step 2: Sharp asymptotics around a concentration point

Denote by $\rho_{i,\alpha}$ the maximum radius around which the concentration point $x_{i,\alpha}$ is dominant. We obtain sharp asymptotics for the blow-up on $B_{x_{i,\alpha}}(\rho_{i,\alpha})$:

Step 2: Sharp asymptotics around a concentration point

Denote by $\rho_{i,\alpha}$ the maximum radius around which the concentration point $x_{i,\alpha}$ is dominant. We obtain sharp asymptotics for the blow-up on $B_{x_{i,\alpha}}(\rho_{i,\alpha})$:

Proposition

- $\rho_{i,\alpha} \underset{\alpha \to +\infty}{\longrightarrow} 0$

Step 2: Sharp asymptotics around a concentration point

Denote by $\rho_{i,\alpha}$ the maximum radius around which the concentration point $x_{i,\alpha}$ is dominant. We obtain sharp asymptotics for the blow-up on $B_{x_{i,\alpha}}(\rho_{i,\alpha})$:

Proposition

- $\rho_{i,\alpha} \underset{\alpha\to+\infty}{\longrightarrow} 0$
- $\nabla\tau_\alpha(x_{i,\alpha}) \underset{\alpha\to+\infty}{\longrightarrow} 0$ *and* $\nabla f_\alpha(x_{i,\alpha}) \underset{\alpha\to+\infty}{\longrightarrow} 0$ *(localisation),*

Step 2: Sharp asymptotics around a concentration point

Denote by $\rho_{i,\alpha}$ the maximum radius around which the concentration point $x_{i,\alpha}$ is dominant. We obtain sharp asymptotics for the blow-up on $B_{x_{i,\alpha}}(\rho_{i,\alpha})$:

Proposition

- $\rho_{i,\alpha} \underset{\alpha \to +\infty}{\longrightarrow} 0$
- $\nabla \tau_\alpha(x_{i,\alpha}) \underset{\alpha \to +\infty}{\longrightarrow} 0$ *and* $\nabla f_\alpha(x_{i,\alpha}) \underset{\alpha \to +\infty}{\longrightarrow} 0$ *(localisation),*
- $u_\alpha \sim \mu_{i,\alpha}^{\frac{n-2}{2}} \left(\mu_{i,\alpha}^2 + d_g(x_{i,\alpha}, x)^2\right)^{-\frac{n-2}{2}}$ *in* $C^0(B_{x_{i,\alpha}}(\rho_{i,\alpha}))$,

Step 2: Sharp asymptotics around a concentration point

Denote by $\rho_{i,\alpha}$ the maximum radius around which the concentration point $x_{i,\alpha}$ is dominant. We obtain sharp asymptotics for the blow-up on $B_{x_{i,\alpha}}(\rho_{i,\alpha})$:

Proposition

- $\rho_{i,\alpha} \underset{\alpha \to +\infty}{\longrightarrow} 0$
- $\nabla \tau_\alpha(x_{i,\alpha}) \underset{\alpha \to +\infty}{\longrightarrow} 0$ *and* $\nabla f_\alpha(x_{i,\alpha}) \underset{\alpha \to +\infty}{\longrightarrow} 0$ *(localisation),*
- $u_\alpha \sim \mu_{i,\alpha}^{\frac{n-2}{2}} \left(\mu_{i,\alpha}^2 + d_g(x_{i,\alpha}, x)^2\right)^{-\frac{n-2}{2}}$ *in* $C^0(B_{x_{i,\alpha}}(\rho_{i,\alpha}))$*, where*

  $\mu_{i,\alpha} = \left(\max_{B_{x_{i,\alpha}}(\rho_{i,\alpha})} u_\alpha\right)^{-\frac{2}{n-2}}$,

Step 2: Sharp asymptotics around a concentration point

Denote by $\rho_{i,\alpha}$ the maximum radius around which the concentration point $x_{i,\alpha}$ is dominant. We obtain sharp asymptotics for the blow-up on $B_{x_{i,\alpha}}(\rho_{i,\alpha})$:

Proposition

- $\rho_{i,\alpha} \underset{\alpha \to +\infty}{\longrightarrow} 0$
- $\nabla \tau_\alpha(x_{i,\alpha}) \underset{\alpha \to +\infty}{\longrightarrow} 0$ *and* $\nabla f_\alpha(x_{i,\alpha}) \underset{\alpha \to +\infty}{\longrightarrow} 0$ *(localisation),*
- $u_\alpha \sim \mu_{i,\alpha}^{\frac{n-2}{2}} \left(\mu_{i,\alpha}^2 + d_g(x_{i,\alpha}, x)^2\right)^{-\frac{n-2}{2}}$ *in* $C^0(B_{x_{i,\alpha}}(\rho_{i,\alpha}))$*, where* $\mu_{i,\alpha} = \left(\max_{B_{x_{i,\alpha}}(\rho_{i,\alpha})} u_\alpha\right)^{-\frac{2}{n-2}}$,
- $|\mathcal{L}_g W_\alpha|_g \sim \mu_{i,\alpha}^{n-1} \rho_\alpha^{-n} \left(\mu_{i,\alpha}^2 + d_g(x_{i,\alpha}, x)^2\right)^{-\frac{n-1}{2}}$ *in* $C^0(B_{x_{i,\alpha}}(\rho_{i,\alpha}))$.

Step 3: Interactions between different concentration points and contradiction

The contradiction will be obtained by analyzing the interactions between the different concentration points.

Step 3: Interactions between different concentration points and contradiction

The contradiction will be obtained by analyzing the interactions between the different concentration points. We let:

$$d_\alpha = \min_{i \neq j \leq N_\alpha} d_g\left(x_{i,\alpha}, x_{j,\alpha}\right)$$

be the minimum distance between two concentration points.

Step 3: Interactions between different concentration points and contradiction

The contradiction will be obtained by analyzing the interactions between the different concentration points. We let:

$$d_\alpha = \min_{i \neq j \leq N_\alpha} d_g\left(x_{i,\alpha}, x_{j,\alpha}\right)$$

be the minimum distance between two concentration points. We will prove that both $d_\alpha \to 0$ and $d_\alpha \not\to 0$.

Step 3: Interactions between different concentration points and contradiction

The contradiction will be obtained by analyzing the interactions between the different concentration points. We let:

$$d_\alpha = \min_{i \neq j \leq N_\alpha} d_g\left(x_{i,\alpha}, x_{j,\alpha}\right)$$

be the minimum distance between two concentration points. We will prove that both $d_\alpha \to 0$ and $d_\alpha \not\to 0$.

The local analysis of the previous step showed that:

$$d_\alpha \to 0$$

as $\alpha \to +\infty$. This heavily relied on the assumptions on the physics data of the theorem.

Step 3: Interactions between different concentration points and contradiction

The contradiction will be obtained by analyzing the interactions between the different concentration points. We let:

$$d_\alpha = \min_{i \neq j \leq N_\alpha} d_g \left(x_{i,\alpha}, x_{j,\alpha} \right)$$

be the minimum distance between two concentration points. We will prove that both $d_\alpha \to 0$ and $d_\alpha \not\to 0$.

The local analysis of the previous step showed that:

$$d_\alpha \to 0$$

as $\alpha \to +\infty$. This heavily relied on the assumptions on the physics data of the theorem.

On the other side, if $d_\alpha \to 0$, it is possible to prove that (u_α, W_α) blows-up with the same speed at each concentration point.

Step 3: Interactions between different concentration points and contradiction

The contradiction will be obtained by analyzing the interactions between the different concentration points. We let:

$$d_\alpha = \min_{i \neq j \leq N_\alpha} d_g\left(x_{i,\alpha}, x_{j,\alpha}\right)$$

be the minimum distance between two concentration points. We will prove that both $d_\alpha \to 0$ and $d_\alpha \not\to 0$.

The local analysis of the previous step showed that:

$$d_\alpha \to 0$$

as $\alpha \to +\infty$. This heavily relied on the assumptions on the physics data of the theorem.

On the other side, if $d_\alpha \to 0$, it is possible to prove that (u_α, W_α) blows-up with the same speed at each concentration point. Therefore, u_α and W_α cannot be exactly equivalent to a rescaled limiting profile at first order.

Step 3: Interactions between different concentration points and contradiction

The contradiction will be obtained by analyzing the interactions between the different concentration points. We let:

$$d_\alpha = \min_{i \neq j \leq N_\alpha} d_g\left(x_{i,\alpha}, x_{j,\alpha}\right)$$

be the minimum distance between two concentration points. We will prove that both $d_\alpha \to 0$ and $d_\alpha \not\to 0$.

The local analysis of the previous step showed that:

$$d_\alpha \to 0$$

as $\alpha \to +\infty$. This heavily relied on the assumptions on the physics data of the theorem.

On the other side, if $d_\alpha \to 0$, it is possible to prove that (u_α, W_α) blows-up with the same speed at each concentration point. Therefore, u_α and W_α cannot be exactly equivalent to a rescaled limiting profile at first order. Hence $d_\alpha \not\to 0$.

Final Remark: Elliptic Stability as a tool

We described elliptic stability here as a way to justify that the solutions constructed by the conformal method are well-behaved.

Final Remark: Elliptic Stability as a tool

We described elliptic stability here as a way to justify that the solutions constructed by the conformal method are well-behaved. It is in general a powerful tool to address existence issues via the a priori estimates it provides.

Final Remark: Elliptic Stability as a tool

We described elliptic stability here as a way to justify that the solutions constructed by the conformal method are well-behaved. It is in general a powerful tool to address existence issues via the a priori estimates it provides.

- It was used to prove the generic existence of two solutions of the Einstein-Lichnerowicz equation in the focusing case (P. '14)

Final Remark: Elliptic Stability as a tool

We described elliptic stability here as a way to justify that the solutions constructed by the conformal method are well-behaved. It is in general a powerful tool to address existence issues via the a priori estimates it provides.

- It was used to prove the generic existence of two solutions of the Einstein-Lichnerowicz equation in the focusing case (P. '14)
- It has very recently been used to prove involved existence results in the non-variational setting of the so-called volumetric drift method (Vâlcu, '19).

Final Remark: Elliptic Stability as a tool

We described elliptic stability here as a way to justify that the solutions constructed by the conformal method are well-behaved. It is in general a powerful tool to address existence issues via the a priori estimates it provides.

- It was used to prove the generic existence of two solutions of the Einstein-Lichnerowicz equation in the focusing case (P. '14)
- It has very recently been used to prove involved existence results in the non-variational setting of the so-called volumetric drift method (Vâlcu, '19). There elliptic stability is required in order to apply topological methods.

Thank you for your attention!