The hyperbolic positive energy theorem

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joint work with P.T. Chruściel (the slides too!)

Positive energy for asymptotically flat manifolds
space-dimension n

Theorem (Lohkamp 2016; Schoen, Yau, 2017)

The ADM mass of n-dimensional asymptotically flat Riemannian manifolds, $n \geq 3$, is non-negative, and vanishes only for Euclidean space.

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Asymptotically hyperbolic metrics

Geometric formulae for total energy (Ashtekar Romano 1992; Herzlich 2015; Chruściel, Barzegar, Höerzinger 2017), space-dimension n

$$\mathbf{g} \to_{r\to\infty} \overline{\mathbf{g}} = -V^2 dt^2 + V^{-2} dr^2 + r^2 d\Omega^2 \,, \qquad V = r^2 + 1 \,.$$

- For any Killing vector X of $\overline{\mathbf{g}}$ we have

$$H_b(X, \mathscr{S}) = \frac{1}{16(n-2)\pi} \lim_{R\to\infty} \int_{t=0, r=R} X^\nu Z^\xi W^{\alpha\beta}{}_{\nu\xi} dS_{\alpha\beta} \,,$$

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• Riemannian version, asymptotically hyperbolic Riemannian metrics g, $\mathrm{R}^i{}_j$ is the Ricci tensor of g:

$$H_b(X, \mathscr{S}) = -\frac{1}{16(n-2)\pi} \lim_{R\to\infty} \int_{r=R} X^0 V Z^j (\mathrm{R}^i{}_j - \frac{\mathrm{R}}{n} \delta^i_j) dS_i \,.$$

Theorem (with P. T. Chruściel, arXiv:1901.05263)

The energy-momentum vector of n-dimensional AH manifolds (M, g) with $R(g) \geq -n(n-1)$, $n \geq 3$, is timelike future pointing or null.

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Theorem (Isenberg, Lee & Stavrov 2010, with P.T. Chruściel JDG 2018)

Given two asymptotically hyperbolic vacuum initial data sets one can construct a new one by making a connected sum at the conformal boundary at infinity. The construction can be localised by a Carlotto-Schoen type hyperbolic gluing.

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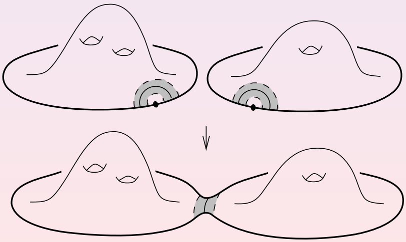

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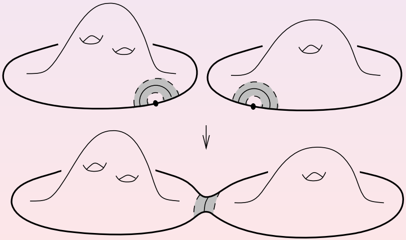

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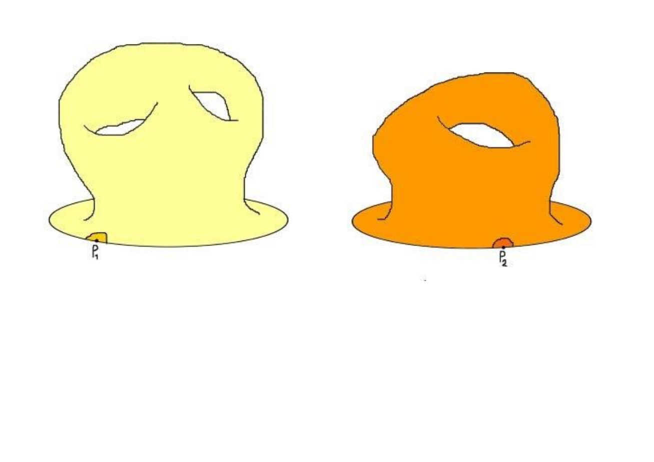
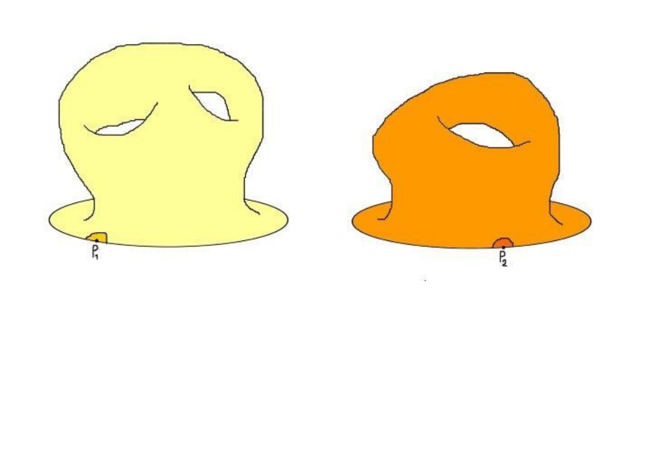


Glue the hyperbolic metric on a "small" part near infinity

$m_1 \to m_1^\epsilon$, $m_2 \to m_2^\epsilon$. Use *hyperbolic isometries* such that :



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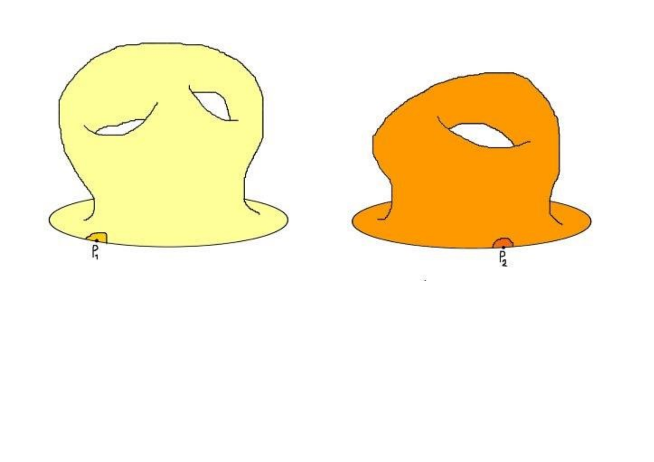
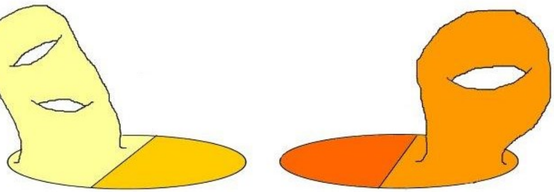


Glue the hyperbolic metric on a ”small” part near infinity $m_1 \to m_1^\epsilon$, $m_2 \to m_2^\epsilon$. Use *hyperbolic isometries* such that :

The "small $\mathbb{H}^n$" part becames a "half $\mathbb{H}^n$".



$m_1^\epsilon \to \Lambda_\epsilon^1 m_1^\epsilon$ and $m_2^\epsilon \to \Lambda_\epsilon^2 m_2^\epsilon$. *Cut and paste :*

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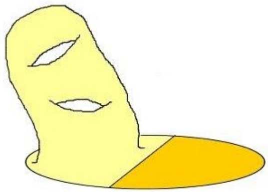

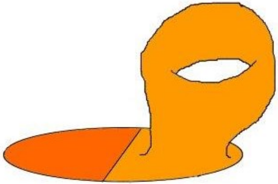

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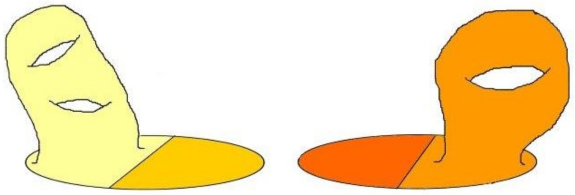

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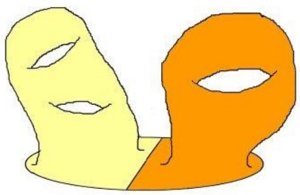
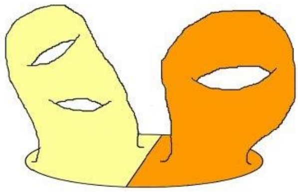


$$m^{\epsilon} = \Lambda^1_{\epsilon} m^{\epsilon}_1 + \Lambda^2_{\epsilon} m^{\epsilon}_2.$$



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The mass aspect function of n-dimensional asymptotically hyperbolic Riemannian manifolds, $3 \le n \le 7$, cannot be negative (everywhere).

$$g = x^{-2}\left(dx^2 + \left(h_{AB}(y^C) + x^n \mu_{AB}(y^C)\right) dy^A dy^B + \text{ lower order}\right) ,$$

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$$m_0 = \int_{S^{n-1}} \Theta \, d^{n-1} y \, ,$$

where the mass aspect function is defined as

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The mass aspect function of n-dimensional asymptotically hyperbolic Riemannian manifolds, $3 \leq n \leq 7$, cannot be negative (everywhere).

1. Uses a deformation argument independent of dimension, and a positivity theorem valid for $3 \leq n \leq 7$
2. Different story if conformal infinity is not spherical
3. One can use the Lohkamp – Schoen-Yau theorem to remove the dimension assumption [Chruściel-D 2019]
4. This version uses another deformation argument for $n \geq 4$
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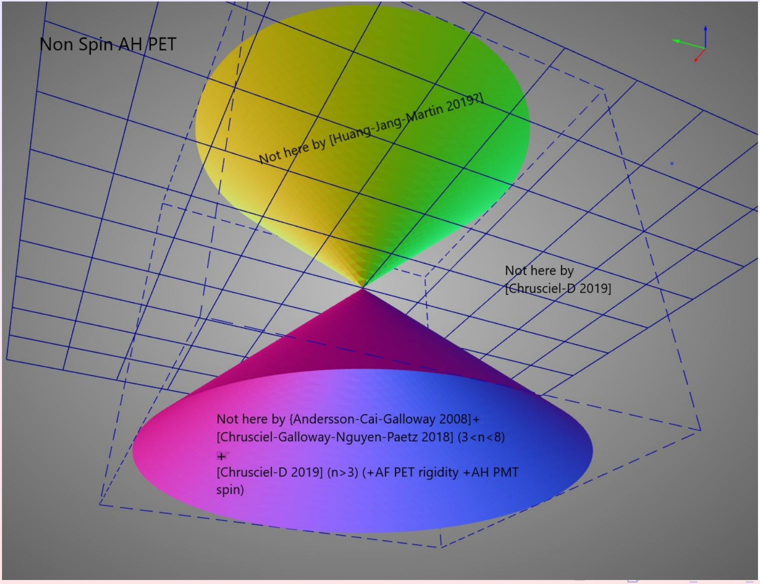
Non Spin AH PET
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[Chrusciel-Galloway-Nguyen-Paetz 2018] (3<n<8)
+
[Chrusciel-D 2019] (n>3) (+AF PET rigidity +AH PMT
spin)

m can not be timelike past pointing :

If m timelike past pointing, $\exists g = hyp$ outside compact set and $R(g) \geq -n(n-1) = 2\Lambda$ [ACG2008].

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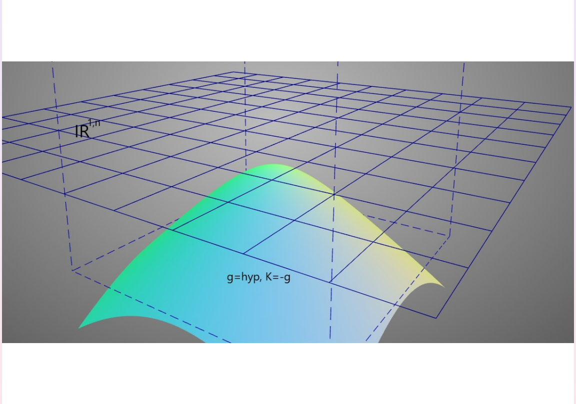

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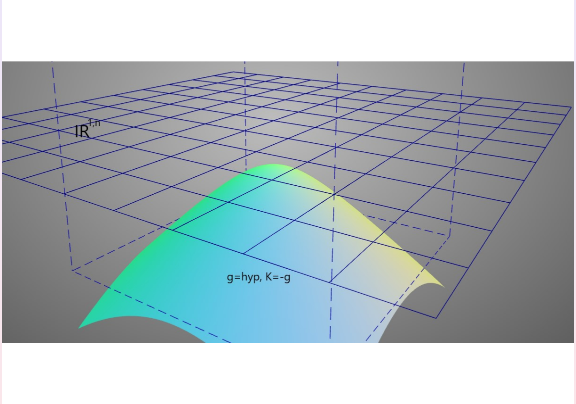

Let $g = g$ and $K = -g$: hyperboloïd outside of a compact set and $R(g) - |K|_g^2 + (Tr_g K)^2 \geq 0$.

Change in $\mathbb{R}^{1,n}$ the hyperoloïd near infinty by an horizontal hyperplane (just a graph, $R(g) - |K|_g^2 + (Tr_g K)^2 = 0$ there)

rigid part of *AF PET* $\Rightarrow (M, g, K)$ slice in $\mathbb{R}^{1,n} \Rightarrow M$ spin $\Rightarrow$ (M, g) hyperbolic space (*rigid part of spin AH PMT*) $\Rightarrow m = 0$.

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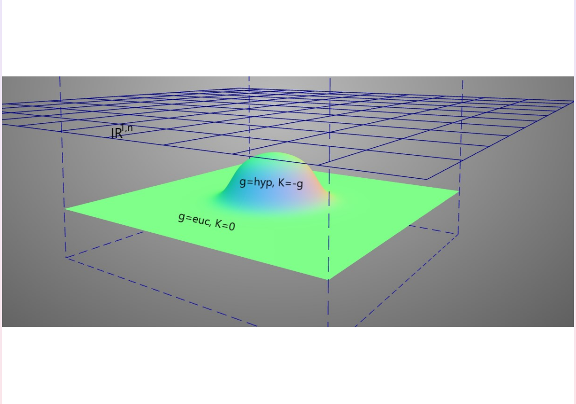

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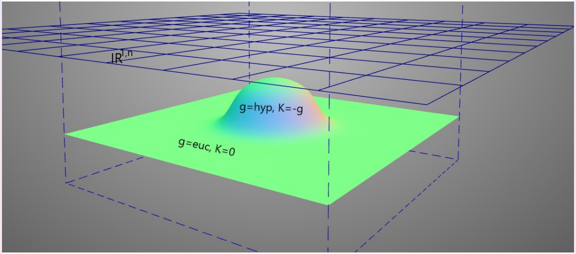

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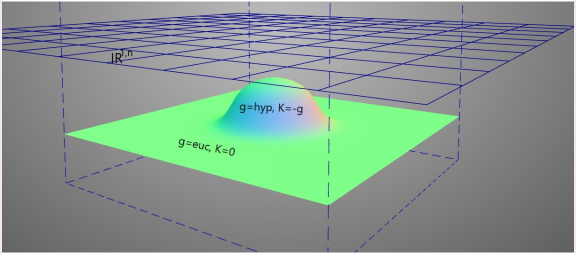

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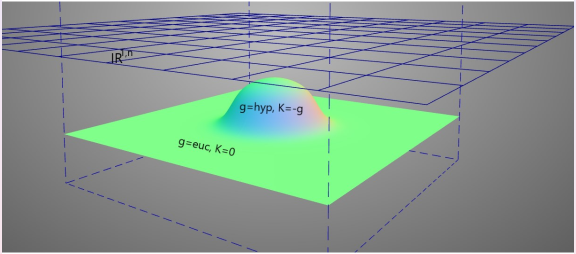

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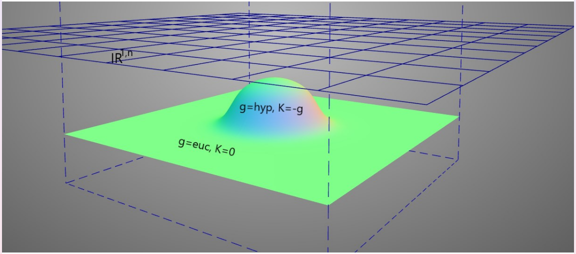

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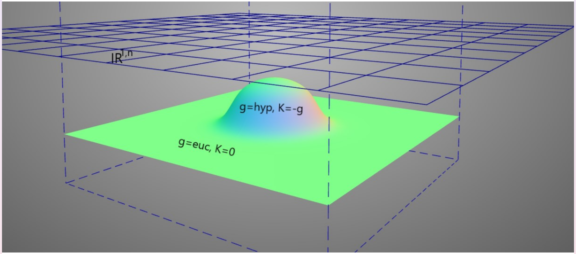

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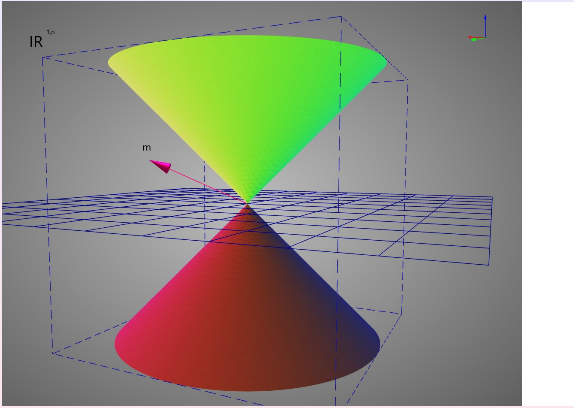
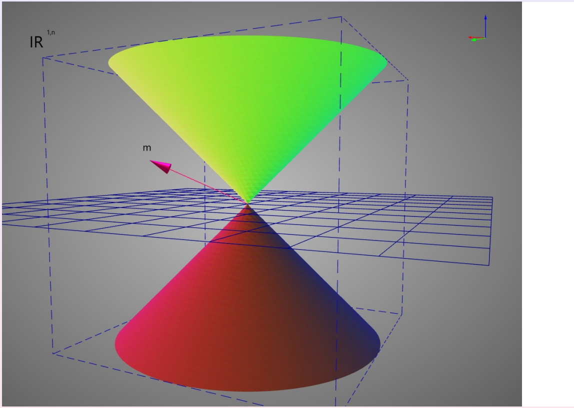


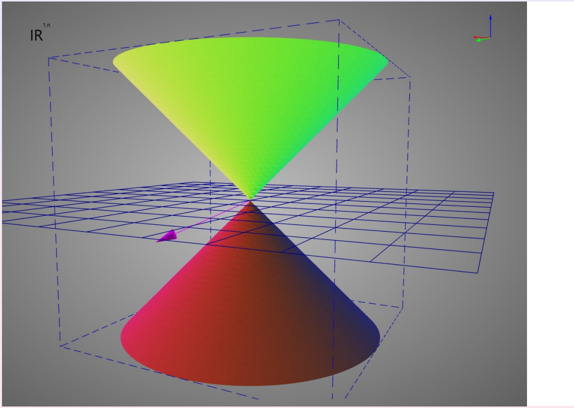
Use an *hyperbolic isometry* such that :

If *m* is *spacelike* with positive time component



Use an *hyperbolic isometry* such that :

m becames *spacelike* with negative time component:



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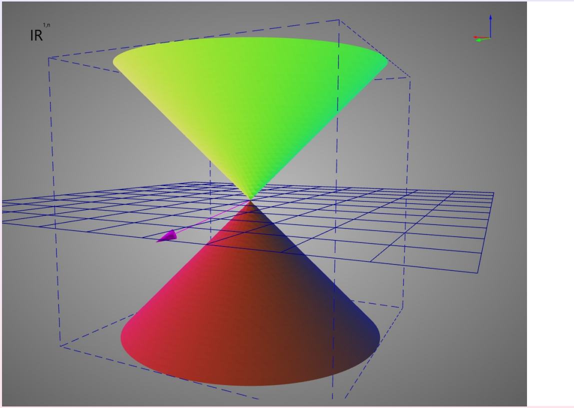

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Positive energy for asymptotically hyperbolic manifolds

m can not be spacelike :

If $m^1 = (m_0^1, \vec{m}^1)$ and $m^2 = (m_0^1, -\vec{m}^1) = R_\pi m^1$, $m_0^1 < 0$.

If $m = m^1 + m^2 = (m_0^1, \vec{m}^1) + (m_0^1, -\vec{m}^1) = (2m_0^1, \vec{0})$. Not the case but we have :

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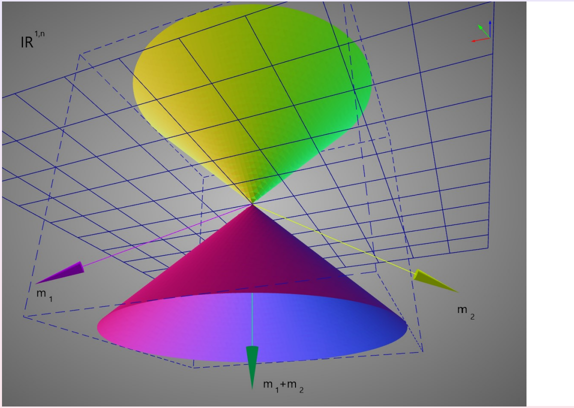

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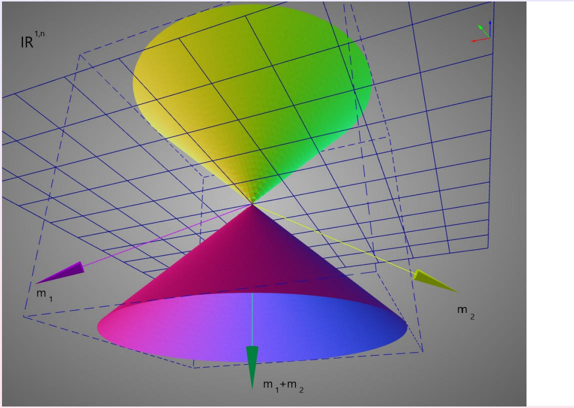

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\begin{aligned}
m^\varepsilon &= \Lambda^1_\varepsilon m^{1,\varepsilon} + \Lambda^2_\varepsilon m^{2,\varepsilon} = \Lambda^1_\varepsilon m^{1,\varepsilon} + R_\pi \Lambda^1_\varepsilon m^{1,\varepsilon} \\
&= (\Lambda^1_\varepsilon + R_\pi \Lambda^1_\varepsilon)(m^1 + m^{1,\varepsilon} - m^1) \\
&= \gamma_\varepsilon \Big(2(\underbrace{m^1_0 - v_\varepsilon m^1_1}_{<0}, \vec{0}) + \\
&\qquad \gamma_\varepsilon^{-1} \Lambda^1_\varepsilon \big(\underbrace{(1 + (\Lambda^1_\varepsilon)^{-1} R_\pi \Lambda^1_\varepsilon)(m^{1,\varepsilon} - m^1)}_{=:(*)} \big) \Big) .
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|(*)| &\equiv |(1 + (\Lambda^1_\varepsilon)^{-1} R_\pi \Lambda^1_\varepsilon)(m^{1,\varepsilon} - m^1)| \\
&\leq C(1 + \varepsilon^{-2})|m^{1,\varepsilon} - m^1| \leq C^2 o(\varepsilon^{\frac{n}{2}-2}) ,
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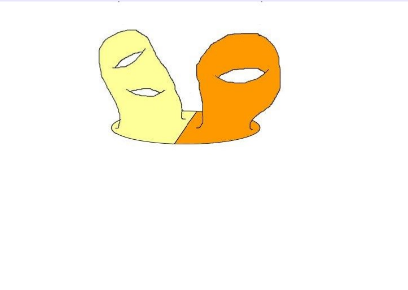

THANK YOU !